

## Weekly Exercises 3

Room: 02.09.023

Friday, 6.5.2016, 09:00-11:00

Submission deadline: Wednesday, 4.5.2016, 14:00, Room 02.09.023

### Theory: Strong convexity, Lipschitz continuity and subgradient descent (12 Points)

**Exercise 1** (4 Points). Prove the following theorem: If  $E \in \mathcal{S}_{m,L}^{1,1}(\mathbb{R}^n)$ , then for any  $u, v \in \mathbb{R}^n$  we have

$$\langle \nabla E(u) - \nabla E(v), u - v \rangle \geq \frac{mL}{m+L} \|u - v\|^2 + \frac{1}{m+L} \|\nabla E(u) - \nabla E(v)\|^2$$

**Exercise 2** (4 Points). Compute the subdifferentials of the following convex functions:

1.  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f(x) = \|x\|_1$ .
2.  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f(x) = \|x\|_2$ .
3.  $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$  with  $f(X) = \|X\|_{2,1} := \sum_{i=1}^m \|x_i\|_2$ , where  $x_i \in \mathbb{R}^n$  is the  $i$ -th column of  $X$ .

**Exercise 3** (4 Points). Let  $f \in \mathbb{R}^n$ . Show that the  $\ell_1$ -norm proximity operator of  $f$  defined as the solution  $u$  of the convex optimization problem

$$\arg \min_{u \in \mathbb{R}^n} \frac{1}{2\lambda} \|u - f\|^2 + \|u\|_1,$$

is given as

$$u \in \mathbb{R}^n, \quad u_i := \begin{cases} f_i + \lambda & \text{if } f_i < -\lambda \\ 0 & \text{if } f_i \in [-\lambda, \lambda] \\ f_i - \lambda & \text{if } f_i > \lambda. \end{cases}$$

Hint: Note that the above optimization problem is decoupled in the sense that one can look for the individual entries  $u_i$  of the optimal  $u$  separately.

## Programming: TV- $\ell_1$ -denoising

(12 Points)

**Exercise 4** (12 Points). Denoise the noisy input image  $f$ , given in the file `fish_saltpepper.png` by minimizing the following robust denoising energy:

$$E(u) = \frac{\lambda}{2} \|u - f\|_1 + \|Du\|_{2,1}$$

with subgradient descent, where  $D$  a finite difference color gradient operator, and the  $\ell_{2,1}$ -norm is defined as on exercise sheet 0.