Convex Optimization for Computer Vision
Lecture: M. Möller and T. Möllenhoff
Exercises: E. Laude
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Weekly Exercises 4

Room: 02.09.023
Friday, 13.5.2016, 09:00-11:00
Submission deadline: Wednesday, 11.5.2016, 14:00, Room 01.09.014

Theory: Projected gradient descent (12 Points)

Exercise 1 (4 Points)**.** Let $A \in \mathbb{R}^{n \times n}$ be orthonormal, meaning that $A^\top A = AA^\top = I$. Let the convex set C be given as

$$C := \{u \in \mathbb{R}^n : \|Au\|_\infty \leq 1\}.$$

Compute a formula for the projection onto C given as

$$\Pi_C(v) := \arg\min_{u \in \mathbb{R}^n} \|u - v\|_2^2, \quad \text{s.t. } u \in C.$$

Hint: Show that the ℓ_2-norm of a vector is invariant under a multiplication with an orthonormal matrix A, meaning that $\|u\|_2 = \|Au\|_2$.

Exercise 2 (8 Points)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be coercive, differentiable and let ∇f be locally Lipschitz continuous i.e. for each $x \in \mathbb{R}^n$ there exists $\epsilon > 0$ and a constant $L_\epsilon > 0$, such that ∇f is L_ϵ-Lipschitz on $B_\epsilon(x)$.

1. Give an example of a function $f : \mathbb{R}^n \to \mathbb{R}$ that meets the above assumptions, but for which ∇f is not Lipschitz continuous. (With a proof).
2. Let f be an arbitrary function meeting the above assumptions. Show that for any $\alpha \geq \min_x f(x)$ there exists a constant L_α such that f is L_α-smooth on the sublevel set $S_\alpha := \{x : f(x) \leq \alpha\}$.
   Hint: You can use the topological definition of compactness: Any subset $X \subset \mathbb{R}^n$ is called compact (closed and bounded in the Euclidean case that we consider) if each of its open covers has a finite subcover (see `https://en.wikipedia.org/wiki/Compact_space`). One possible open cover of X is the union of all ϵ-balls with any ϵ: $\bigcup_{x \in X} B_\epsilon(x)$.
3. Show that $x^+ := x - \tau \nabla f(x) \in S_\alpha$ if $\tau < \frac{1}{L_\alpha}$ and $x \in S_\alpha$.
4. Conclude that for each initialization x_0, there exists a τ_{x_0} such that gradient descent with the constant step size τ_{x_0} converges.

Exercise 3 (4 Points)**.** Let C_i, $1 \leq i \leq n$ be a family of closed convex sets such that

$$\bigcap_{1 \leq i \leq n} C_i \neq \emptyset.$$

Show that the problem of finding an element u^* in the intersection

$$u^* \in \bigcap_{1 \leq i \leq n} C_i$$

can be formulated as the following optimization problem:

$$u^* \in \arg \min_{u \in \bigcap_{i \in \mathcal{I}} C_i} \sum_{\substack{j \notin \mathcal{I} \\ 1 \leq j \leq n}} d^2(u, C_j),$$

where $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$ can be arbitrary (including the empty set) and $d(z, X)$ is the distance of a point z to the closed convex set X defined as

$$d(z, X) := \min_{x \in X} \|x - z\|_2.$$

Programming: SUDOKU (12 Points)

Exercise 4 (12 Points)**.** Solve the SUDOKUs given in the files `exampleSudoku1.mat` and `exampleSudoku2.mat` with projected gradient descent. For that you need to find a point

$$u^* \in \bigcap_{1 \leq i \leq n+m+1} C_i$$

where the convex sets in the intersection are given as

$$C_i := \{u \in \mathbb{R}^{729} : \langle a_i, u \rangle = 1\}, \quad 1 \leq i \leq n,$$

$$C_i := \{u \in \mathbb{R}^{729} : u_j = 1,\ j \in \mathcal{B}\}, \quad n+1 \leq i \leq n+m,$$

and $\mathcal{B}$ is the set of indexes corresponding to the known numbers and

$$C_{n+m+1} := \{u \in \mathbb{R}^{729} : u_j \in [0, 1],\ \forall\, 1 \leq j \leq 729\}.$$

For a more precise definition of the constraint sets see lecture.
Solve the programming assignment in the spirit of exercise 3 using the following two partitions of the indexes $\{1, 2, \ldots, n+m+1\}$:

1. $\mathcal{I}_1 := \{n+m+1\}$ and
2. $\mathcal{I}_2 := \{n+1, n+2, \ldots, n+m+1\}$,

and plot the resulting energy decays.
Hint: Show that for the linear constraint sets C_i, $1 \leq i \leq n$ the distance $d(z, C_i)$ of a point z to the set C_i is equal to $|\langle a_i, z \rangle - 1|$.