Convex Optimization for Computer Vision
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Summer Semester 2016

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# Weekly Exercises 7

Room 01.09.014

Friday, 10.6.2016, 09:00-11:00

Submission deadline: Wednesday, 8.6.2016, 14:15, Room: 02.09.023

## Theory: (12+2 Points)

**Exercise 1** (6 Points)**.** Let $F$ and $G$ be proper, closed, convex functions, let $E(u) := G(u) + F(Ku)$ have a minimizer, and let there be a $u \in \mathrm{ri}(\mathrm{dom}(G))$ such that $Ku \in \mathrm{ri}(\mathrm{dom}(F))$. We define

$$PD(u,p) = G(u) + \langle Ku, p\rangle - F^*(p).$$

Show that there exists a point $(\tilde{u}, \tilde{p})$ such that the (unrestricted) primal-dual gap

$$\mathcal{G}(u,p) = \sup_{p'} PD(u,p') - \inf_{u'} PD(u',p)$$

is zero, i.e. $\mathcal{G}(\tilde{u}, \tilde{p}) = 0$.

**Exercise 2** (2 Points)**.** Prove that the algorithm

$$\begin{aligned}
u^{k+1} &= \mathrm{prox}_{\tau G}(u^k - \tau K^* \bar{p}^k),\\
p^{k+1} &= \mathrm{prox}_{\sigma F^*}(p^k + \sigma K u^{k+1}),\\
\bar{p}^{k+1} &= 2p^{k+1} - p^k.
\end{aligned} \tag{PDHG*}$$

converges, and the limit of the $u^k$ is a minimizer of $G(u) + F(Ku)$ (with the same assumptions on $F$, $G$, and $K$ as in the previous exercise).

*Hint: Show that* (PDHG*) *is equivalent to an algorithm we discussed in the lecture applied to a reformulated problem!*

**Exercise 3** (6 Points)**.** Derive explicit formulas for the update equations for solving the following problems with the (PDHG) algorithm:

- TV-$\ell^1$ denoising, i.e.
$$\min_u \|u - f\|_1 + \alpha \|Ku\|_{2,1}$$
for a linear operator $K : \mathbb{R}^{n\times m\times 3} \to \mathbb{R}^{nm\times 2c}$ (e.g. being the finite difference color gradient operator).

- Image segmentation, i.e.

$$\min_u \iota_\Delta(u) + \iota_{\geq 0}(u) + \langle u, f\rangle + \alpha\|Ku\|_{2,1}$$

  for a linear operator $K : \mathbb{R}^{n\times m\times 3} \to \mathbb{R}^{nmc\times 2}$ (e.g. being the finite difference gradient operator).

- Sparse recovery, i.e.

$$\min_u \frac{1}{2}\|Au - f\|^2 + \alpha\|u\|_1$$

  for $A : \mathbb{R}^n \to \mathbb{R}^m$ being a linear operator.

# Programming: TV-$\ell^1$-denoising revisited (12 Points)

**Exercise 4** (12 Points)**.** Consider the salt-and-pepper removal algorithm from Exercise 4 on Sheet 3 again, i.e.

$$u^* = \arg\min_u \ \|u - f\|_1 + \alpha\|Du\|_{2,1}, \tag{1}$$

where $K$ is a finite difference color gradient operator.

On Sheet 3 you implemented subgradient descent. Now implement the (PDHG) algorithm to compute the argument that minimizes (1). Plot the decay of energy of the subgradient descent and the (PDHG) method for at least two different choices of stepsizes $\tau$ and $\sigma$ on the `fish_saltpepper.png` image.

*Hint: To make sure the step size restriction of (PDHG) is met, the MATLAB function "normest" can be useful.*