Convex Optimization for Computer Vision
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# Weekly Exercises 9

Room 01.09.014

Friday, 24.6.2016, 09:00-11:00

Submission deadline: Wednesday, 22.6.2016, 14:15, Room: 02.09.023

## Theory: Monotone Operators (12+6 Points)

**Exercise 1** (4 points)**.** Let the matrix $L \in \mathbb{R}^{n\times n}$ be positive definite and the operator $T \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be monotone. Show that

$$L^{-\top} \circ T \circ L^{-1}$$

is monotone.

**Exercise 2** (4 Points)**.** Let $E : \mathbb{R}^n \to \mathbb{R}$ be differentiable. As we have seen in the lecture gradient descent can be formulated in the framework of monotone operators as

$$u^{k+1} = (I - \tau\nabla E)u^k$$

1. Suppose $E$ is $m$-strongly convex and $L$-smooth. Show that $I-\tau\nabla E$ is Lipschitz with
$$L_{GM} = \max\{|1-\tau m|, |1-\tau L|\}.$$
2. Let $E$ be $L$-smooth. Show that $I - \nabla E$ is Lipschitz with
$$L_{GM} = \max\{1, |1-\tau L|\}.$$

Hint: You may assume that $E$ is 2 times continuously differentiable.

**Exercise 3** (6 Points)**.** Consider the primal-dual hybrid gradient method (PDHG) in operator form:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial G & -K^* \\ K & \partial F^* \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} u^{k+1} \\ p^{k+1} \end{pmatrix}}_{=:z^{k+1}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}I & K^* \\ K & \frac{1}{\sigma}I \end{pmatrix}}_{=:M} \underbrace{\begin{pmatrix} u^{k+1}-u^k \\ p^{k+1}-p^k \end{pmatrix}}_{=z^{k+1}-z^k}$$

- Assume that a saddle point $z^* := (u^*, p^*)$, i.e. an element with $0 \in Az^*$ exists. Show that the PDHG iterates are Fejer-monotone, i.e.
$$\|z^{k+1} - z^*\|_M \leq \|z^k - z^*\|_M,$$
where $\|v\|_M^2 = \langle v, Mv\rangle$ for a positive definite matrix $M$.
*Hint: Write the above inclusion with respect to $z_e^{k+1} = z^{k+1} - z^*$ and $z_e^k = z^k - z^*$, and take the inner product with $z_e^{k+1}$.*

- Assume that $G$ and $F^*$ are $\gamma$-strongly convex, but $\tau\sigma\|K\|^2 = 1$ such that $M$ is positive semi-definite only. Show that $z^k \to z^*$, where $z^*$ is the (unique) saddle point ($0 \in Az^*$).
  *Hint: $(A - \gamma I)$ is still a monotone operator!*

**Exercise 4** (4 Points)**.** Let $G : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be proper, closed and convex. Let $K \in \mathbb{R}^{m\times n}$. Show the following identity.

$$\text{prox}_{\sigma(G^* \circ -K^*)}(v) = v + \sigma K \cdot \arg\min_u G(u) + \frac{\sigma}{2} \left\| Ku + \frac{v}{\sigma} \right\|^2.$$

# Programming: Deblurring revisited (12 Points)

**Exercise 5** (12 Points)**.** Given a blurry and noisy input image $f$, reconstruct a sharper image $u^*$ by solving the following optimization problem

$$u^* = \arg\min_u \frac{1}{2} \|k * u - f\|^2 + \alpha \|Du\|_{2,1}, \tag{1}$$

with ADMM. For details on how to deal with the convolution cf. sheet 6. Since we consider the non-blind setting, i.e. a known blur kernel $k$, we obtain a blurry and noisy input image by convolving the image with the kernel $k$ and adding a small amount of Gaussian noise.