Weekly Exercise 2

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Probability theory (12 Points)

Exercise 1 (σ**-algebra**, 4 points)**.**

a) Let $\mathcal{A}$ be a σ-algebra over Ω. Show that $\Omega \in \mathcal{A}$.

b) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, such that

$$A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A} \quad \text{and}$$
$$A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A} \; .$$

Show that

$$\emptyset \in \mathcal{A} \Leftrightarrow \mathcal{A} \neq \emptyset \Leftrightarrow \Omega \in \mathcal{A}.$$

c) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra. Show that $\mathcal{A}$ is *closed under intersections*, i.e.

$$A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \in \mathcal{A} \; .$$

Solution. a) Note that $\emptyset \in \mathcal{A}$ is satisfied by definition. Moreover $\bar{\emptyset} = \Omega \setminus \emptyset = \Omega$. By applying the definition of σ-algebras (see Property 2), one can get

$$\emptyset \in \mathcal{A} \Rightarrow \bar{\emptyset} = \Omega \in \mathcal{A} \; .$$

b) By applying a), it is easy to see that $\emptyset \in \mathcal{A} \Leftrightarrow \Omega \in \mathcal{A}$ holds.

If $\mathcal{A} \neq \emptyset$, then $\exists A \in \mathcal{A}$ such that $\bar{A} \in \mathcal{A}$ (see Property 2). Because of Property 3, $A \cup \bar{A} = \Omega \in \mathcal{A}$. Therefore $\mathcal{A} \neq \emptyset \Leftrightarrow \Omega \in \mathcal{A}$ also holds.

c)

$$A_1, A_2 \in \mathcal{A} \overset{\text{Property 2}}{\Rightarrow} \bar{A}_1, \bar{A}_2 \in \mathcal{A} \overset{\text{Property 3}}{\Rightarrow} \bar{A}_1 \cup \bar{A}_2 \in \mathcal{A} \overset{\text{Property 2}}{\Rightarrow} \overline{\overline{A_1} \cup \overline{A_2}} = A_1 \cap A_2 \in \mathcal{A}.$$

Exercise 2 (**Probability**, 4 points)**.** In order to express his gratitude, Siegfried invites Eduard to a pub for a couple of beers. There, they start playing a friendly game of darts. The dart board is a perfect disk of radius 10cm. If a dart falls within 1cm of the center, 100 points are scored. If the dart hits the board between 1 and 3cm from the center, 50 points are scored, if it is at a distance of 3 to 5cm 25 points are scored and if it is further away than 5cm 10 points are scored. As Siegfried and Eduard are both quite experienced dart players, they hit the dart board every time. Siegfried places the dart uniformly on the board.

a) Define the probability space $(\Omega, \mathcal{A}, P)$.

b) What is the probability that Siegfried scores 100 points on one throw?

c) What is the probability of him scoring 50 points on one throw?

d) Eduard is very focused and thus twice more likely to hit the inner 4cm part of the board than the outer region. On each region, the dart arrives uniformly. Answer the previous questions now for Eduard's throw.

Solution. a)

$$\begin{aligned}
\Omega &= \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 10\} , \\
\mathcal{A} &= \left\{A \subset \Omega \mid \int_\Omega \chi_A(x) \, \mathrm{d}x \text{ exists.}\right\} , \\
P &: \mathcal{A} \to [0, 1], P(A) = \frac{\int_\Omega \chi_A(x) \mathrm{d}x}{100\pi} .
\end{aligned}$$

b)

$$P(\text{the score is } 100) = \frac{\pi}{100\pi} = 0.01 .$$

c)

$$P(\text{the score is } 50) = \frac{9\pi - \pi}{100\pi} = \frac{8}{100} = 0.08 .$$

d) Let $C \subset \Omega$ denote the event of the dart hitting the *inner 4cm region*. Then

$$P(C) + P(\Omega \setminus C) = 1 \quad \Rightarrow \quad 2P(\Omega \setminus C) + P(\Omega \setminus C) = 1 ,$$

thus $P(C) = \frac{2}{3}$.

Let $A \subset \Omega$ denote the event of the dart hitting the inner-most region and $B \subset \Omega$ the event of the dart hitting the $1 - 3cm$ region. Then

$$\begin{aligned}
P(A) &= P(A \mid C)P(C) = \frac{\pi}{16\pi} \frac{2}{3} = \frac{1}{24} . \\
P(B) &= P(B \mid C)P(C) = \frac{9\pi - \pi}{16\pi} \frac{2}{3} = \frac{1}{3} .
\end{aligned}$$

Exercise 3 (Bayes' rule, 1 point**).** Let A, B, C be *events*. Assuming $P(B \mid C) \neq 0$, prove that

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C) \cdot P(A \mid C)}{P(B \mid C)} .$$

Solution. By applying the definition of conditional probability, we get

$$\begin{aligned}
\frac{P(B \mid A \cap C) \cdot P(A \mid C)}{P(B \mid C)} &= \frac{P(A \cap B \cap C) \cdot P(A \cap C)}{P(A \cap C) \cdot P(C)} \frac{P(C)}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \\
&= P(A \mid B \cap C) .
\end{aligned}$$

Exercise 4 (**Bayes' rule**, 3 points)**.** Siegfried the ornithologist does a study on the green-speckled swallow. Since he has a huge collection of bird photographs he wants to find all images depicting a green-speckled swallow. Due to it's distinctive features it is an easy task for Eduard, Siegfried's friend and computer vision scientist, to program a green-speckled swallow detector that marks all images containing such a bird. Unfortunately the detector does not work perfectly. If the image contains a green-speckled swallow the detector marks it correctly with a chance of 99.5%. If the image does not contain a green-speckled swallow the detector marks it correctly with a chance of 99.3%. The bird is also very rare: If we randomly draw an image from the collection, there is only a chance of 0.001% that the image contains a green-speckled swallow.

a) Do a formal modeling of the experiment. How does the discrete probability space look like?

b) What is the probability that a green-speckled swallow is on a given image, if the detector gives a positive answer?

c) What is the probability that a green-speckled swallow is on a given image, if the detector gives a negative answer?

Solution. a) The probability space $(\Omega, \mathcal{A}, P)$ is defined by

$$\begin{aligned}
&\Omega = \{(s,+),(s,-),(n,+),(n,-)\},\\
&\mathcal{A} = \{A \subset \Omega\},\\
&P : \mathcal{A} \to [0,1],
\end{aligned}$$

where the symbols $s, n, +, -$ have the following meaning:

s	: image contains a green-speckled swallow
n	: image does not contain a green-speckled swalow
$+$	: detector reports a green-speckled swallow
$-$	: detector reports no green-speckled swallow

b) Let us introduce the following events

$A = \{(s,+),(s,-)\}$	: the image contains a green-speckled swallow
$B = \{(s,+),(n,+)\}$	: the detector reports positive answer
$C = \{(s,-),(n,-)\}$	: the detector reports negative answer
$D = \{(n,+),(n,-)\}$	: the image does not contain green-speckled swallow

From the text we know that

$$P(B \mid A) = 0.995\ , \quad P(C \mid D) = 0.993 \quad \text{and} \quad P(A) = 0.00001\ .$$

Note that $\bar{A} = D$ and $\bar{B} = C$. Therefore

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A) \cdot P(A)}{P(B)} \\ &= \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid D) \cdot P(D)} \\ &= \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + (1 - P(C \mid D)) \cdot (1 - P(A))} = 1.41942 \cdot 10^{-3} \, . \end{aligned}$$

c)

$$\begin{aligned} P(A \mid C) &= \frac{P(C \mid A) \cdot P(A)}{P(C)} \\ &= \frac{(1 - P(B \mid A)) \cdot P(A)}{P(C \mid A) \cdot P(A) + P(C \mid D) \cdot P(D)} \\ &= \frac{(1 - P(B \mid A)) \cdot P(A)}{(1 - P(B \mid A)) \cdot P(A) + P(C \mid D) \cdot (1 - P(A))} = 5.03529 \cdot 10^{-8} \, . \end{aligned}$$