Weekly Exercise 5

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Graphical models (2 points)

Exercise 1 (Factor graph, 2 points). Let G be a factor graph given by a Markov random field consisting of N^2 binary variables, representing the pixels of a $N \times N$ image. For each pixel there is a unary potential, and there are pairwise potentials according to the 8-connected neighbourhood.

a) Draw the factor graph for $N = 3$.

b) How many factors (of each type) are there, depending on N?

Minimum cut and maximum flow (10 points)

Exercise 2 (Flow, 3 points). Show that the following two definitions are equivalent.

a) Let $(\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network with non-negative edge weights. A function $f : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is called a flow if it satisfies the following properties:

   i) Capacity constraint: $f(i, j) \leq c(i, j)$ for all $i, j \in \mathcal{V}$.
   ii) Skew-symmetry: $f(i, j) = -f(j, i)$ for all $i, j \in \mathcal{V}$.
   iii) Flow conservation: $\sum_{j \in \mathcal{V}} f(i, j) = 0$ for all $i \in \mathcal{V} \setminus \{s, t\}$.

b) Let $(\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network with non-negative edge weights. A function $f : \mathcal{E} \to \mathbb{R}^+$ is called a flow if it satisfies the following two properties:

   i) $f(i, j) \leq c(i, j)$ for all $(i, j) \in \mathcal{E}$.
   ii) For all $i \in \mathcal{V} \setminus \{s, t\}$

$$\sum_{(i,j) \in \mathcal{E}} f(i, j) = \sum_{(j,i) \in \mathcal{E}} f(i, j) \ .$$

Exercise 3 (Flow, 3 points). Let $G = (\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network, and let f be a flow in G. Show that the following equalities hold:

a) For all $X \subseteq \mathcal{V}$, we have $f(X, X) = 0$.

b) For all $X, Y \subseteq \mathcal{V}$, we have $f(X, Y) = -f(Y, X)$.

c) For all $X, Y, Z \subseteq \mathcal{V}$ with $X \cap Y = \varnothing$, we have the sums

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z) \quad \text{and} \quad f(Z, X \cup Y) = f(Z, X) + f(Z, Y) \ .$$

Exercise 4 (**Edmonds–Karp algorithm**, 4 points)**.** Solve the maximum flow problem corresponding to the flow network in Figure 1 by applying the Edmonds–Karp algorithm. Find the minimum $s - t$ cut as well. Draw the residual network and the flow graph for each iteration.

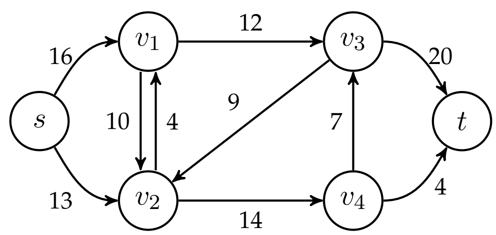

Figure 1: A flow network.