Weekly Exercise 5

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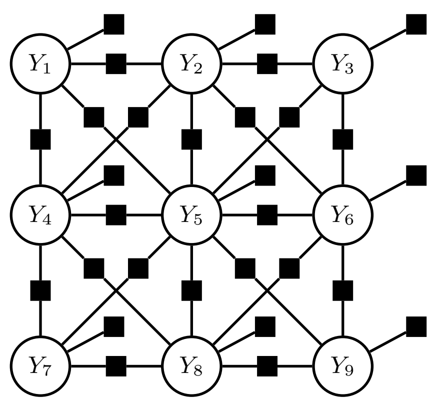
Graphical models (2 points)

Exercise 1 (Factor graph, 2 points). Let G be a factor graph given by a Markov random field consisting of N^2 binary variables, representing the pixels of a $N \times N$ image. For each pixel there is a unary potential, and there are pairwise potentials according to the 8-connected neighbourhood.

a) Draw the factor graph for $N = 3$.

b) How many factors (of each type) are there, depending on N?

Solution. a) The factor graph for $N = 3$ is given as follows:



b) The total number of factors is $N^2 + 4(N-1)^2 + 2(N-1)$.

Minimum cut and maximum flow (10 points)

Exercise 2 (Flow, 3 points). Show that the following two definitions are equivalent.

a) Let $(\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network with non-negative edge weights. A function $f : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is called a flow if it satisfies the following properties:

   i) Capacity constraint: $f(i,j) \leq c(i,j)$ for all $i, j \in \mathcal{V}$.

   ii) Skew-symmetry: $f(i,j) = -f(j,i)$ for all $i, j \in \mathcal{V}$.

   iii) Flow conservation: $\sum_{j \in \mathcal{V}} f(i,j) = 0$ for all $i \in \mathcal{V} \setminus \{s, t\}$.

b) Let $(\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network with non-negative edge weights. A function $f : \mathcal{E} \to \mathbb{R}^+$ is called a flow if it satisfies the following two properties:

   i) $f(i, j) \leq c(i, j)$ for all $(i, j) \in \mathcal{E}$.

   ii) For all $i \in \mathcal{V} \setminus \{s, t\}$
$$\sum_{(i,j)\in\mathcal{E}} f(i, j) = \sum_{(j,i)\in\mathcal{E}} f(j, i) \ .$$

Solution. $a) \Rightarrow b)$ Suppose we are given a flow f satisfying the definition given in a). Let us introduce $f' : \mathcal{E} \to \mathbb{R}^+$ such that $f'(i, j) := \max(0, f(i, j))$ for all $(i, j) \in \mathcal{E}$.

i) $0 \leq f'(i, j) = \max(0, f(i, j)) \leq \max(0, c(i, j)) = c(i, j)$ for all $(i, j) \in \mathcal{E}$.

ii) For all $i \in \mathcal{V} \setminus \{s, t\}$, we have
$$\begin{aligned}\sum_{(i,j)\in\mathcal{E}} f'(i, j) &= \sum_{(i,j)\in\mathcal{E},\, f(i,j)\geq 0} f(i, j) \\ &= \sum_{(i,j)\in\mathcal{E},\, f(j,i)\leq 0} f(i, j) \\ &= \sum_{(j,i)\in\mathcal{E},\, f(j,i)\geq 0} f(j, i) \\ &= \sum_{(j,i)\in\mathcal{E}} f'(j, i) \ .\end{aligned}$$

$b) \Rightarrow a)$ Suppose we are given a flow f satisfying the definition given in b). Let us introduce $f' : \mathcal{E} \to \mathbb{R}$ such that
$$f'(i, j) = \begin{cases} f(i, j) & \text{if } (i, j) \in \mathcal{E} \\ -f(i, j) & \text{if } (j, i) \in \mathcal{E} \ . \end{cases}$$

i) $c(i, j) \geq f(i, j) \geq f'(i, j)$ for all $(i, j) \in \mathcal{E}$.

ii) By definition $f'(i, j) = -f'(j, i)$ for all $(i, j) \in \mathcal{E}$.

iii) For all $i \in \mathcal{V} \setminus \{s, t\}$, we have
$$\sum_{(i,j)\in\mathcal{E}} f'(i, j) = \sum_{(i,j)\in\mathcal{E}} f(i, j) = \sum_{(j,i)\in\mathcal{E}} f(j, i) = -\sum_{(i,j)\in\mathcal{E}} f'(i, j) \ ,$$

which completes the proof.

Exercise 3 (Flow, 3 points**).** Let $G = (\mathcal{V}, \mathcal{E}, c, s, t)$ be a flow network, and let f be a flow in G. Show that the following equalities hold:

a) For all $X \subseteq \mathcal{V}$, we have $f(X, X) = 0$.

b) For all $X, Y \subseteq \mathcal{V}$, we have $f(X, Y) = -f(Y, X)$.

c) For all $X, Y, Z \subseteq \mathcal{V}$ with $X \cap Y = \varnothing$, we have the sums

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z) \quad \text{and} \quad f(Z, X \cup Y) = f(Z, X) + f(Z, Y) .$$

Solution. a) Assume that b) is already held. Then $f(X, X) = -f(X, X)$ for all $X \subseteq \mathcal{V}$, which means that $f(X, X) = 0$.

b) For all $X, Y \subseteq \mathcal{V}$, we have

$$f(X, Y) = \sum_{a \in X} \sum_{b \in Y} f(a, b) = \sum_{a \in X} \sum_{b \in Y} -f(b, a) = -\sum_{b \in Y} \sum_{a \in X} f(b, a) = -f(Y, X) .$$

c) Suppose that $X, Y \subseteq \mathcal{V}$ and $X \cap Y = \emptyset$. Then for any $Z \subseteq \mathcal{V}$, we have

$$f(X \cup Y, Z) = \sum_{a \in X \cup Y} \sum_{b \in Z} f(a, b) = \sum_{a \in X} \sum_{b \in Z} f(a, b) + \sum_{a \in Y} \sum_{b \in Z} f(a, b) = f(X, Z) + f(Y, Z) .$$

One can prove similarly the second equality, too.

Exercise 4 (**Edmonds–Karp algorithm**, 4 points)**.** Solve the maximum flow problem corresponding to the flow network in Figure **??** by applying the Edmonds–Karp algorithm. Find the minimum $s - t$ cut as well. Draw the residual network and the flow graph for each iteration.

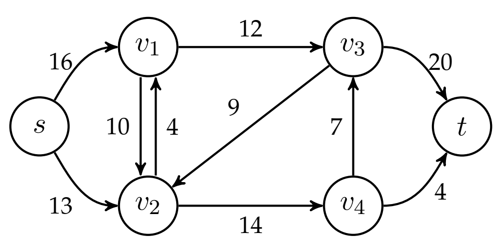


Figure 1: A flow network.

Solution. The maximum flow problem can be solved in three iterations using the Edmonds–Karp algorithm. The residual network and the flow for each iteration are shown below.

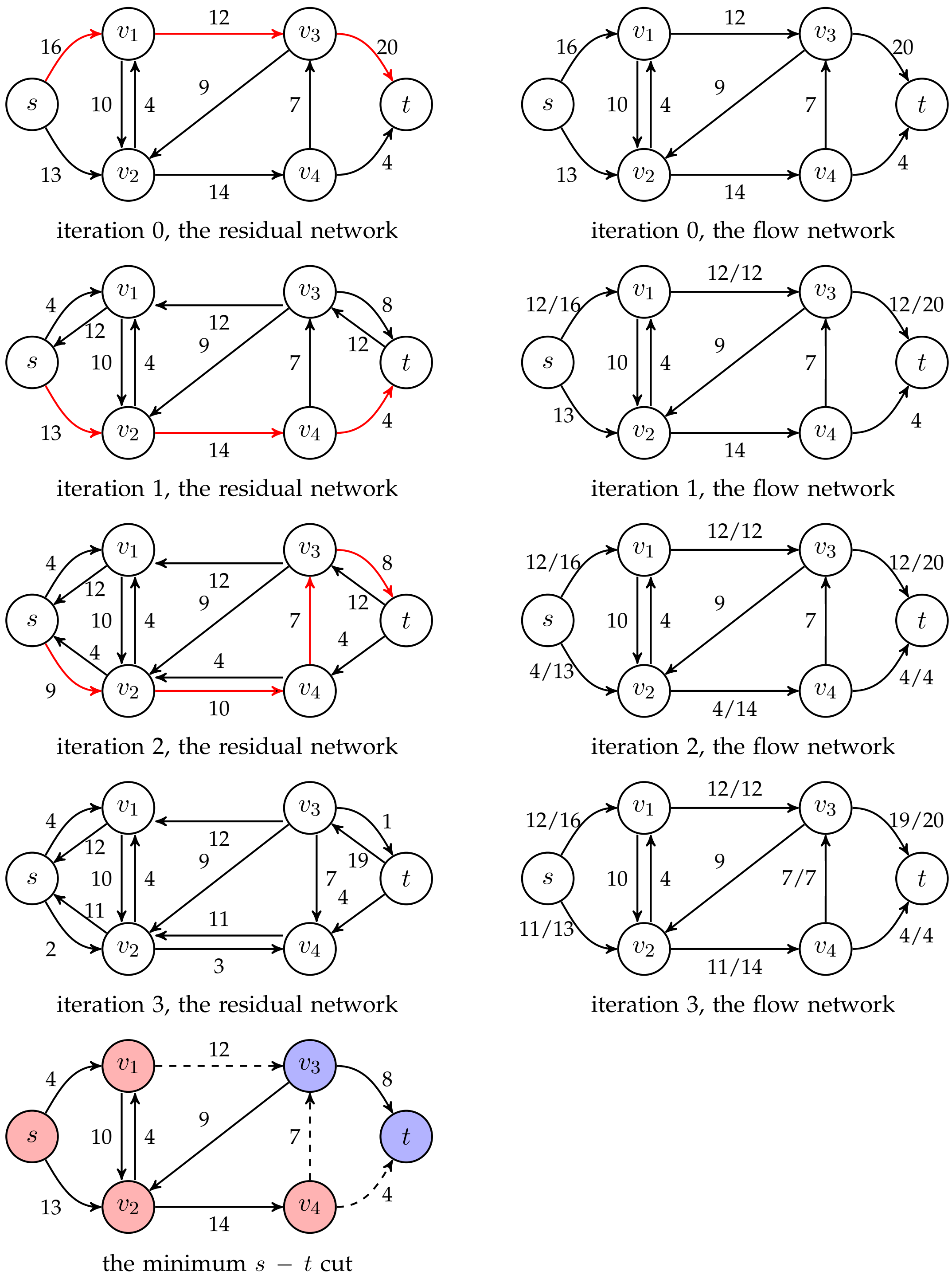


Figure 2: Solution to the maximum flow problem by applying the Edmonds–Karp algorithm. The shortest path that is applied as an augmenting path in the next iteration is marked red. The minimum cut is marked with dashed lines and the two partitions of the nodes are marked red and blue, respectively.