Weekly Exercise 8

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June 07, 2016 (submission deadline: June 14, 2016)

Branch and Bound (2 points)

Exercise 1 (**Lower bound**, 2 Points). For a finite set Ω, consider the following segmentation energy function $E : \mathbb{B}^n \to \mathbb{R}$:

$$E(\mathbf{y}) = \min_{\omega \in \Omega} C(\omega) + \sum_{i=1}^{n} f_i(\omega) y_i + \sum_{i=1}^{n} b_i(\omega)(1 - y_i) + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} w_{ij}(\omega) |y_i - y_j|, \quad (1)$$

with $C : \Omega \to \mathbb{R}$, $f_i : \Omega \to \mathbb{R}$, $b_i : \Omega \to \mathbb{R}$, $w_{ij} : \Omega \to \mathbb{R}$. Prove the following lower bound:

$$\begin{aligned} E(\mathbf{y}) \geq \left(\min_{\omega \in \Omega} C(\omega) \right) + \sum_{i=1}^{n} \left(\min_{\omega \in \Omega} f_i(\omega) \right) y_i + \sum_{i=1}^{n} \left(\min_{\omega \in \Omega} b_i(\omega) \right) (1 - y_i) \\ + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}(i)} \left(\min_{\omega \in \Omega} w_{ij}(\omega) \right) |y_i - y_j| =: \ell(\mathbf{y}, \Omega). \end{aligned} \quad (2)$$

Remark: This shows that $E^* = \min_{\mathbf{y}} E(\mathbf{y}) \geq \min_{\mathbf{y}} \ell(\mathbf{y}, \Omega) = L(\Omega)$ and $L(\Omega)$ is a lower bound for the global optimum. Note that the lower bound $L(\Omega)$ fulfills three important properties which make it applicable for branch and bound optimization methods:

1. **Monotonicity:** $\Omega_1 \subset \Omega_2 \Rightarrow L(\Omega_1) \geq L(\Omega_2)$.
2. **Computability:** Evaluating $L(\Omega)$ for some given Ω corresponds to minimizing a submodular quadratic pseudo-Boolean function.
3. **Tightness:** For $|\Omega| = 1$, i.e. $\Omega = \{\omega\}$ we have $L(\{\omega\}) = \min_{\mathbf{y}} E(\mathbf{y})$.

Solution. One can prove the inequality from the mathematical deduction,

$$
\begin{aligned}
E(\mathbf{y}) &= \min_{\omega\in\Omega} C(\omega) + \sum_{i=1}^{n} f_i(\omega) y_i + \sum_{i=1}^{n} b_i(\omega)(1-y_i) + \sum_{i=1}^{n}\sum_{j\in\mathcal{N}(i)} w_{ij}(\omega)|y_i - y_j| \\
&\geq \min_{\omega\in\Omega} C(\omega) + \min_{\omega\in\Omega}\sum_{i=1}^{n} f_i(\omega) y_i + \min_{\omega\in\Omega}\sum_{i=1}^{n} b_i(\omega)(1-y_i) \\
&\quad + \min_{\omega\in\Omega}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}(i)} w_{ij}(\omega)|y_i - y_j| \\
&= \min_{\omega\in\Omega} C(\omega) + \sum_{i=1}^{n}\min_{\omega\in\Omega} f_i(\omega) y_i + \sum_{i=1}^{n}\min_{\omega\in\Omega} b_i(\omega)(1-y_i) \\
&\quad + \sum_{i=1}^{n}\sum_{j\in\mathcal{N}(i)}\min_{\omega\in\Omega} w_{ij}(\omega)|y_i - y_j| \\
&= \left(\min_{\omega\in\Omega} C(\omega)\right) + \sum_{i=1}^{n}\left(\min_{\omega\in\Omega} f_i(\omega)\right) y_i + \sum_{i=1}^{n}\left(\min_{\omega\in\Omega} b_i(\omega)\right)(1-y_i) \\
&\quad + \sum_{i=1}^{n}\sum_{j\in\mathcal{N}(i)}\left(\min_{\omega\in\Omega} w_{ij}(\omega)\right)|y_i - y_j| =: \ell(\mathbf{y},\Omega).
\end{aligned} \tag{3}
$$

Or, since y_i, $1 - y_i$ and $|y_i - y_j|$ are all positive for $\mathbf{y} \in \mathbb{B}^n$ and we have by definition from the minimum,

$$
C(\omega') \geq \min_{\omega\in\Omega} C(\omega),\ \forall \omega' \in \Omega, \tag{4}
$$

the inequality holds.

Programming (7 points)

Exercise 2 (**Branch-and-Mincut**[1], 7 Points)**.** In this exercise we apply the branch and bound method from the lecture to find a *global minimizer* of a discrete version of the celebrated Chan-Vese[2] segmentation energy function:

$$
\begin{aligned}
E(\mathbf{y}, \{c_f, c_b\}) = \mu &\sum_{i=1}^{n}\sum_{j\in\mathcal{N}(i)} |y_i - y_j| \\
&+ \sum_{i=1}^{n}\left(\nu + \lambda_1 (I_i - c_f)^2\right) y_i + \sum_{i=1}^{n} \lambda_2 (I_i - c_b)^2 (1 - y_i).
\end{aligned} \tag{5}
$$

Here I denotes a gray-scale input image with n pixels, i.e. at every pixel $1 \leq i \leq n$ we have $I_i \in [0, 255]$. The variable $\omega = (c_f, c_b) \in \Omega = [0, 255]^2$ denotes the mean intensity of foreground respectively the background of the segmentation $x \in \mathbb{B}^n$.

[1] V. Lempitsky, A. Blake, C. Rother, Image Segmentation by Branch-and-Mincut, ECCV 2008

[2] T. Chan, L. Vese: Active contours without edges. Trans. Image Process., 10(2), 2001.

Compute a global minimizer of (5) using the branch and bound best-first tree search. The search space Ω is the rectangle $[0, 255]^2$. In your implementation, you can keep a sorted queue of rectangles Ω_i, and every iteration remove the rectangle with the smallest lower bound and split it into two smaller rectangles along the longest edge. As a lower bound on (5) use the bound (2) dervied in the theoretical exercise. You can use `chanvese_global.cpp` as a starting point.

Figure 1: The figure shows the input image and a global minimizer of (5) for parameters $\lambda_1 = \lambda_2 = 0.0001$, $\mu = 1$, $\nu = 0.1$. The optimal foreground and background colors were found as $c_f^* = 81$ and $c_b^* = 167$.