Weekly Exercise 10

Dr. Csaba Domokos and Lingni Ma
Technische Universität München, Computer Vision Group
June 21, 2016 (submission deadline: June 28, 2016)

Mean Field Approximation (4 Points)

Exercise 1 (**Naive Mean Field**, 4 Points)**.** Assume a graphical model $G = (\mathcal{V}, \mathcal{E})$ and consider a factorized distribution in the following form:

$$q(y) = \prod_{i \in \mathcal{V}} q_i(y_i). \tag{1}$$

a) Show that the marginal distribution of a factor F is given by:

$$\mu_{F,y_F}(q) = q_{N(F)}(y_F) = \prod_{i \in N(F)} q_i(y_i) \ .$$

b) Show that the entropy decomposes as:

$$H(y) = \sum_{i \in \mathcal{V}} H_i(y_i) \ ,$$

in other words,

$$\sum_y p(y) \log p(y) = \sum_{i \in \mathcal{V}} \sum_{y_i \in \mathcal{Y}_i} -q_i(y_i) \log q_i(y_i)$$

Solution.

a) First note that:

$$\begin{aligned}
\sum_{y' \in \mathcal{Y}, y'_1 = y_1} q(y) &= \sum_{y' \in \mathcal{Y}, y'_1 = y_1} \prod_{i \in \mathcal{V}} q_i(y_i) && (2)\\
&= q_1(y_1) \sum_{y_2 \in \mathcal{Y}_2} \dots \sum_{y_k \in \mathcal{Y}_k} q_2(y_2) \dots q_k(y_k) && (3)\\
&= q_1(y_1) \underbrace{\sum_{y_2 \in \mathcal{Y}_2} q_2(y_2)}_{1} \dots \underbrace{\sum_{y_k \in \mathcal{Y}_k} \dots q_k(y_k)}_{1} = q_1(y_1). && (4)
\end{aligned}$$

Similarly, one computes:

$$\begin{aligned}
\mu_{F,y_F}(q) &= \sum_{y' \in \mathcal{Y}, y'_F = y_F} q(y') = \sum_{y' \in \mathcal{Y}, y'_F = y_F} \prod_{i \in \mathcal{V}} q_i(y'_i) && (5)\\
&= \prod_{i \in N(F)} q_i(y_i) \underbrace{\sum_{y_j \in \mathcal{Y}_j, j \in \mathcal{V} \setminus N(F)} \prod_{k \in \mathcal{V} \setminus N(F)} q_k(y_k)}_{=1} = \prod_{i \in N(F)} q_i(y_i). && (6)
\end{aligned}$$

b) First, we show that given a set of independent events, the entropy of the joint event can be calculated by the sum of the entropy of each event. This property can be shown with two random variables x, y that are independent of each other. We have,

$$\begin{aligned}
H(xy) &= \sum_x \sum_y -p(xy) \log p(xy) \\
&= \sum_x \sum_y -p(x)p(y)\big(\log p(x) + \log p(y)\big) \\
&= \sum_x \sum_y -p(x)p(y) \log p(x) + \sum_x \sum_y -p(x)p(y) \log p(y) \\
&= \sum_x -p(x) \log p(x) \sum_y p(y) + \sum_x p(x) \sum_y -p(y) \log p(y) \\
&= \sum_x -p(x) \log p(x) + \sum_y -p(y) \log p(y) \\
&= H(x) + H(y)
\end{aligned}$$

It is obvious, this property can be generalized to the set of arbitrary number of independent random variables. Now, given the assumption (1), y_i are independent $\forall i \in \mathcal{V}$, hence

$$H(y) = \sum_{i \in \mathcal{V}} H(y_i) = \sum_{i \in \mathcal{V}} \sum_{y_i \in \mathcal{Y}_i} -q_i(y_i) \log q_i(y_i)$$

Programming (10 Points)

Exercise 2 (**Multi-class image segmentation with fully connected CRF**, 10 points)**.** Let us consider the multi-class labeling problem for image segmentation. Assuming the label set $\mathcal{L}$, we define the following energy function for $\mathbf{y} \in \mathcal{L}^{\mathcal{V}}$:

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) \quad , \tag{7}$$

on a fully connected CRF model, i.e. $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i < j\}$.

The goal of the exercise is to implement the `naive mean field approximation` algorithm in order to obtain multi-class segmentation results for test images shown in Figure 1. As you have seen in the previous exercises, these images come from a subset of the MSRC image understanding dataset, which contains 21 classes, i.e. $\mathcal{L} = \{1, 2, \ldots, 21\}$.

To define the unary energy functions E_i for $i \in \mathcal{V}$, use the provided `*.c_unary` files. Each test image has its own unary file, specified by the same filename. In each unary file, you can read out a $K \times H \times W$ array of float numbers. The H and W are the image height and width, and $K = 21$ is the number of possible classes. This array contains the 21-class probability distribution for each pixel. We provide the `multilabel_demo.cpp` to demonstrate how to load the unary file and read out the corresponding probability

values. The unary energy functions E_i for all $i \in \mathcal{V}$ are then defined as

$$E_i(y_i = l) = -\log(p_l) \ .$$

To define the pairwise energy E_{ij} for $(i,j) \in \mathcal{E}$, use the contrast sensitive Potts model:

$$E_{ij}(y_i, y_j) = [\![y_i \neq y_j]\!] \left(w_1 \exp\left(-\frac{|p_i - p_j|^2}{2\theta_\alpha^2} - \frac{|I_i - I_j|^2}{2\theta_\beta^2} \right) + w_2 \exp\left(-\frac{|p_i - p_j|^2}{2\theta_\gamma^2} \right) \right),$$

where the parameter are chosen as

$$w_1 = 10, \ w_2 = 3, \ \theta_\alpha = 80, \ \theta_\beta = 13, \ \theta_\gamma = 3 \ .$$

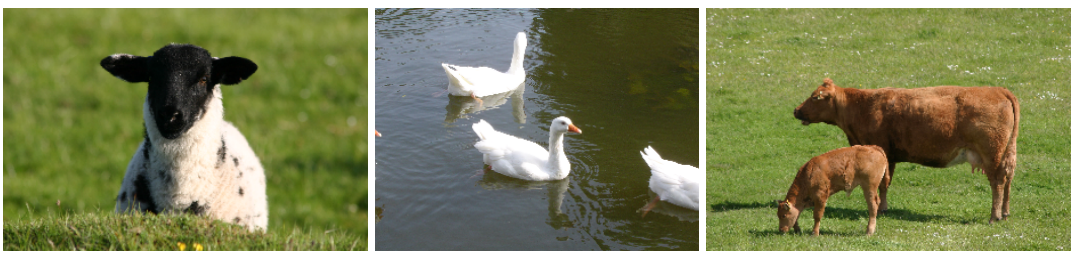

Figure 1: Test images for multi-class image segmentation.