Probabilistic Graphical Models in Computer Vision (IN2329)

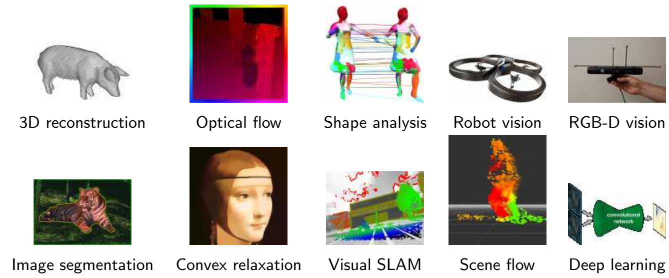
Csaba Domokos

Summer Semester 2015/2016

Announcement: Computer Vision Group *

Inquiries for Bachelor and Master projects are always welcome!

We currently work on the following `research topics`:



Please complete the form: `https://vision.in.tum.de/application`

1. Introduction

Agenda for today's lecture *

1. Administration
2. Overview of the course
3. Introduction to Probability theory
   - Basic definitions
   - Conditional probability, Bayes' rule
   - Independence, conditional independence

Administration

The course: IN2329 *

The course **Probabilistic Graphical Models in Computer Vision** will be organized as follows:

- Lectures: on Tuesdays at 10.00–12.00 in Room 00.13.036
- Tutorials: on Tuesdays at 14.00–16.00 in Room 02.05.014

The tutorials combines *theoretical* and *programming* assignments:

- **Assignment distribution**: Tuesday 11.00–11.15 in Room 00.13.036
- **Theoretical assignment due**: Tuesday 11.00–11.15 in Room 00.13.036
- **Assignment presentation**: Tuesday 14.00–16.00 in Room 02.05.014

Lecturer

Dr. Csaba Domokos (`csaba.domokos@in.tum.de`)

Teaching assistant (TA)

Lingni Ma (`lingni@in.tum.de`)

Feel free to contact us!

April *

April 2016

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
				1	2	3
4	5	6	7	8	9	10
11	12 Lecture 1 Tutorial 1	13	14	15	16	17
18	19 Lecture 2 Tutorial 2	20	21	22	23	24
25	26 Lecture 3 Tutorial 3	27	28	29	30	

May *

May 2016

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
2	3 Lecture 4 Tutorial 4	4	5	6	7	8
9	10 Lecture 5 Tutorial 5	11	12	13	14	15
16	17	18	19	20	21	22
23	24 Lecture 6 Tutorial 6	25	26	27	28	29
30	31 Lecture 7 Tutorial 7					

June *

June 2016

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
		1	2	3	4	5
6	7 Lecture 8 Tutorial 8	8	9	10	11	12
13	14 Lecture 9 Tutorial 9	15	16	17	18	19
20	21 Lecture 10 Tutorial 10	22	23	24	25	26
27	28 Lecture 11 Tutorial 11	29	30			

July *

July 2016

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
				1	2	3
4	5 Lecture 12 Tutorial 12	6	7	8	9	10
11	12 Lecture 13 Tutorial 13	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27 Week of the exam	28	29	30	31

Exam *

- The exam will be oral.
- According to our schedule, the exam will be held in the **last week of July, 25th–29th**.
- Students **need to be registered** prior to the exam: **May, 16th–June, 30th** via TUM online.

Participation at the tutorial:

- **Not mandatory, but highly recommended**:
  Theoretical assignments will help to understand the topics of the lecture.
  Programming assignments will help to apply the theory to practical Computer vision problems.
- **Bonus**: Active students who solve *60% of the assignments* earn a bonus.
  If someone receives a mark between 1.3 and 4.0 in the *final exam*, the mark will be **improved by 0.3 and 0.4, respectively**.
  Note that marks of 1.0 and 5.0 cannot be improved!

Bonus *

To achieve the bonus, the following requirements have to be fulfilled:

Theory

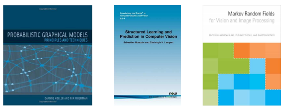
- **60%** of all theoretical assignments have to be solved.
  (Note that submissions happen **only** on Tuesdays at 11.00–11.15)
- The theoretical exercises have to *be presented in front of the class*.
  (The TA randomly selects a student who presents an exercise.)

Programming

- **60%** of all programming assignments have to be presented during the tutorial.
- The programming exercises should *be explained to the TA*.

To promote team work, please form **groups of two or three students** in order to solve and submit the assignments.

Recommended literature & prerequisites *



- D. Koller, N. Friedman. Probabilistic Graphical Models: Principles and Techniques, MIT Press, 2009.
- S. Nowozin, C. H. Lampert. Structured Learning and Prediction in Computer Vision, Foundations and Trends in Computer Graphics and Vision, 2011.
- A. Blake, P. Kohli, C. Rother. Markov Random Fields for Vision and Image Processing, MIT Press, 2011.

In addition, we will mention recent *conference* and *journal papers*.

Prerequisites: the course is intended for *Master students*.

- **Basic Mathematics**: multivariate analysis and linear algebra.
- **Basic Computer Science**: dynamic programming and basic data structures.

Course Page *

On the **internal site** of the `course` page you could access to extra course material:

`https://vision.in.tum.de/teaching/ss2016/lecture_graphical_models/material`

Password: `PGMCV:SS16`

Course materials:

- **Slides for each lecture** (available prior to the lecture)
- **Assignment sheets** (available after the lecture)
- **Solution sheets** (available after the tutorial)

The course page will also be used for extra announcements.

Overview

Overview of the course *

Binary image segmentation *

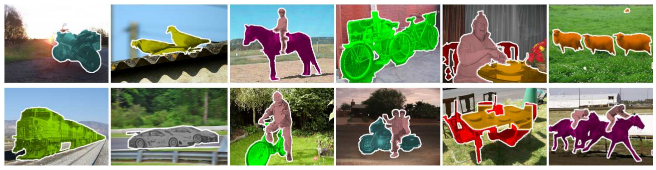
The goal is to give a binary label $y_i \in \mathbb{B} \triangleq \{0,1\}$ for each pixel i, where 0 and 1 mean the *background* (a.k.a. ground) and the *foreground* (a.k.a. figure), respectively.

Input image

Figure–ground segmentation

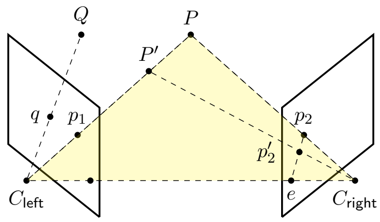
Semantic image segmentation *

The goal is to give a label $y_i \in \mathcal{L} = \{1, 2, \ldots, c\}$ for each pixel i according to its *semantic meanings*.



Exemplar semantic segmentations

Stereo matching *

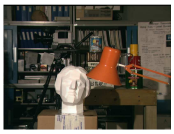


Left view Right view

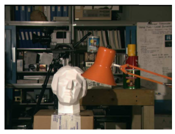
Given two images (i.e. left and right), an observed 2D point p_1 on the *left image*, which corresponds to a 3D point P that is situated on a line in $\mathbb{R}^3$. This line will be observed as a line on the *right image*.
P can be determined based on p_1 and p_2. We assume that the pixels p_1 and p_2, corresponding to P, have similar intensities.

Stereo matching *

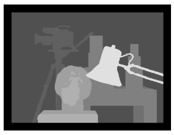
The goal is *to reconstruct 3D points* according to corresponding pixels. Usually we assume **rectified images** (i.e. the directions of the cameras are parallel), which means that the corresponding pixels are situated in horizontal lines.



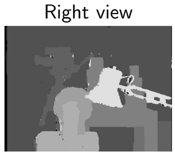
Left view



Right view



Ground truth (depth map)



Result (depth map)

Object detection *

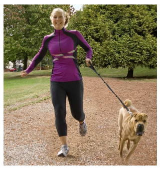
We address the problem of *binary image segmentation*, where we also assume non-local **parameters** that are **known a priori**. For example, one can assume prior knowledge about the **shape** of the foreground.

Exemplar binary segmentation of cars assuming shape prior

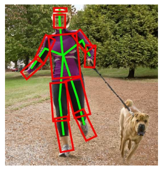
You may realize that we will mainly deal with labelling problems.

Human–pose estimation *

The goal is to recognize an *articulated object* (i.e. human body) with different connecting parts (e.g., head, torso, left arm, right arm, left leg, right leg).



Input image



Human pose estimation

An object is composed of a number of *rigid parts*, where each part is modeled as a **rectangle**. The *connections* encode generic relationships such as "close to", "to the left of".

Probability theory

Reasoning under uncertainty *

We often want to understand a system when we have *imperfect* or *incomplete* information due to, for example, noisy measurement.
There are *two main reasons* why we might **reason under uncertainty**:

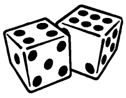
- *Laziness*: modeling every detail of a complex system is *costly*.
- *Ignorance*: we *may not completely understand*.

Probability $P(A)$ refers to a degree of confidence that an event A with uncertain nature will occur.

It is common to assume that $0 \leqslant P(A) \leqslant 1$:

- If $P(A) = 1$, we are certain that A occurs,
- while $P(A) = 0$ asserts that A will not occur.

Experiment, sample space, event



An **experiment** is a (random) process that can be infinitely many times repeated and has a well-defined set of possible **outcomes**. In case of repeated experiments the individual repetitions are also called **trials**.

Example: throwing two "fair dice" (i.e. we assume equally likely chance of landing on any face) with six faces.

The **sample space**, denoted by Ω, is the set of possible outcomes.

Example: $\Omega = \{(i,j) : 1 \leqslant i,j \leqslant 6\}$.

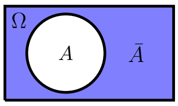
A set of outcomes $A \subseteq \Omega$ is called an **event**. An **atomic event** is an event that contains a single outcome $\omega \in \Omega$.

Example: $A = \{(i,j) : i+j = 11\}$, i.e. the sum of the numbers showing on the top is equal to eleven.

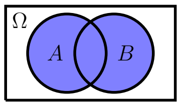
Basic notations

Let A and B be two *events* from an *sample space* Ω. We will use the following notations:

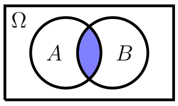
A does not occur: $\bar{A} = \Omega \backslash A$



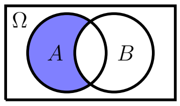
either A or B occur: $A \cup B$



both A and B occur: $A \cap B$



A occurs and B does not: $A \backslash B$



- The $\varnothing$ is called the **impossible event**; and
- Ω is the **sure event**.

Discrete probability space

A *probability space* represents our **uncertainty** regarding an *experiment*.

A triple $(\Omega, \mathcal{A}, P)$ is called a **discrete probability space**, if

- Ω is not empty and **countable** (i.e. $\exists \mathcal{S} \subseteq \mathbb{N}$ such that $|\Omega| = |\mathcal{S}|$),
- $\mathcal{A}$ is the **power set** $\mathcal{P}(\Omega)$ (i.e. the set of all subsets of Ω), and
- $P : \mathcal{A} \to \mathbb{R}$ is a function, called a **probability measure**, with the following properties:
  1. $P(A) \geqslant 0$ for all $A \in \mathcal{A}$
  2. $P(\Omega) = 1$
  3. **σ-additivity** holds: if $A_n \in \mathcal{A}, n = 1, 2, \ldots$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then
$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) \ .$$

The conditions 1-3. are called **Kolmogorov's axioms**.

Example: throwing two "fair dice" *

For this case a *discrete probability space* $(\Omega, \mathcal{A}, P)$ is given by

- **Sample space**: $\Omega = \{(i,j) : 1 \leqslant i,j \leqslant 6\}$.
- $\mathcal{A} = \mathcal{P}(\Omega) = \{\{(1,1)\}, \ldots, \{(1,1),(1,2)\}, \ldots, \{(1,1),(1,2),(1,3)\}, \ldots\}$.
- The **probability measure**
$$P(A) = \frac{|A|}{36} = \frac{k}{36} \ ,$$
where k is the number of *atomic events* in A.

Example: Let A denote the *event* that *"the sum of the numbers showing on the top is equal to eleven"*, that is

$$A = \{(i,j) : i+j = 11\} = \{(5,6),(6,5)\} \ .$$

Hence
$$P(A) = P(\{(5,6),(6,5)\}) = \frac{2}{36} \ .$$

σ-algebra, measure, measure space *

Assume an arbitrary set Ω and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. The set $\mathcal{A}$ is a **σ-algebra over Ω** if the following conditions are satisfied:

1. $\varnothing \in \mathcal{A}$,
2. $A \in \mathcal{A} \Rightarrow \bar{A} \in \mathcal{A}$ (i.e. it is *closed under complementation*),
3. $A_i \in \mathcal{A}$ $(i \in \mathbb{N}) \Rightarrow \bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$ (i.e. it is *closed under countable union*).

It is a consequence of this definition that $\Omega \in \mathcal{A}$ is also satisfied. (See exercise.)

Assume an *arbitrary set* Ω and a *σ-algebra* $\mathcal{A}$ over Ω. A function $P : \mathcal{A} \to [0, \infty]$ is called a **measure** if the following conditions are satisfied:

1. $P(\varnothing) = 0$,
2. P is σ-additive.

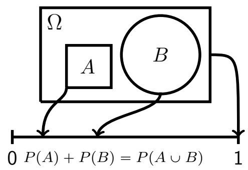
Let $\mathcal{A}$ be a *σ-algebra* over Ω and $P : \mathcal{A} \to [0, \infty]$ is a *measure*. $(\Omega, \mathcal{A})$ is said to be a **measurable space** and the triple $(\Omega, \mathcal{A}, P)$ is called a **measure space**.

Probability space *

A **probability space** is a triple $(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A})$ is a *measurable space*, and P is a *measure* such that $P(\Omega) = 1$, called a **probability measure**.

To summarize:
A triple $(\Omega, \mathcal{A}, P)$ is called **probability space**, if

- the **sample space** Ω is *not empty*,
- $\mathcal{A}$ is a **σ-algebra** over Ω, and
- $P : \mathcal{A} \to \mathbb{R}$ is a function with the following properties:
  1. $P(A) \geqslant 0$ for all $A \in \mathcal{A}$
  2. $P(\Omega) = 1$
  3. **σ-additive**: if $A_n \in \mathcal{A}$, $n = 1, 2, \ldots$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then
$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) \ .$$



Example: throwing a dart *

Suppose a dart is thrown at a round board modeled as a unit circle. The **sample space** contains the location of the dart if it lands in the board only. Hence it is given by

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\} \ .$$

We denote the **area** of an the **event** $A \subseteq \Omega$ by $\mu(A)$, which is defined as the *Riemann-integral* of the **characteristic function of** A

$$\mu(A) := \int_{\Omega} \chi_A(x) \mathrm{d}x \ , \quad \text{where} \quad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \ . \end{cases}$$

The **σ-algebra** $\mathcal{A}$ over Ω is defined as follows

$$\mathcal{A} = \{A \subseteq \Omega : \mu(A) \text{ exists}\} \ .$$

The **probability measure** $P : \Omega \to [0,1]$ is given by $P(A) = \frac{\mu(A)}{\pi}$.

Some simple consequences of the axioms

The following rules are frequently used in applications:

- $P(A) = 1 - P(\Omega \backslash A)$.

  Proof. Note that A and $\Omega \backslash A$ are disjoint.
  $1 = P(\Omega) = P(A \cup (\Omega \backslash A)) = P(A) + P(\Omega \backslash A)$. □

- $P(\varnothing) = 0$.

  Proof. $P(\varnothing) = 1 - P(\Omega \backslash \varnothing) = 1 - P(\Omega) = 1 - 1 = 0$. □

- If $A \subseteq B$, then $P(A) \leqslant P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B) \leqslant P(A) + P(B)$.
- $P(A \backslash B) = P(A) - P(A \cap B)$.
- ...

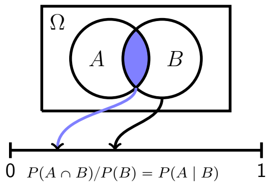
Conditional Probability

Conditional probability

Conditional probability allows us to reason with *partial information*.
If $P(B) > 0$, the **conditional probability of A given B** is defined as

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)} .$$

This is the probability that A occurs, given we have observed B, i.e. we know the experiment's actual outcome will be in B.



Note that the *axioms and rules of probability theory are fulfilled* for the conditional probability. (e.g., $P(A \mid B) = 1 - P(\bar{A} \mid B)$).

Example *

Consider two producing machines creating identical product in a factory. Assume we are given the following table with probabilities

	Machine I	Machine II	
The product is good	0.56	0.41	0.97
The product is waste	0.01	0.02	0.03
	0.57	0.43	1

Question: What is the probability of a product was created by Machine I, when it is good?

Let A denote the event that *"the product was created by Machine I"* and let B denote the event that *"the product is good"*.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.56}{0.97} \approx 0.58 .$$

The chain rule

Starting with the definition of *conditional probability* $P(B \mid A)$ and multiplying by P(A) we get the **product rule**:

$$P(A \cap B) = P(A)P(B \mid A) .$$

The chain rule is given by

$$P(\cap_{i=1}^{n} A_i) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid \cap_{i=1}^{n-1} A_i) . \quad (1)$$

Proof. By induction. For $n = 2$ we get the product rule. Let $n \in \mathbb{N}$ be given and suppose Eq. (1) is true for $k \leqslant n$. Then

$$P(\cap_{i=1}^{n+1} A_i) = P(A_{n+1} \cap (\cap_{i=1}^{n} A_i)) = P(A_{n+1} \mid \cap_{i=1}^{n} A_i)P(\cap_{i=1}^{n} A_i) .$$

□

The chain rule will become important later when we discuss *conditional independence*.

Bayes' rule

By making use of the product rule we can get

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B)} .$$

$P(A \mid B)$ is often called the **posteriori probability**, and $P(B \mid A)$ is called the **likelihood**, and $P(A)$ is called the **prior probability**.

A more general version of **Bayes' rule**, when we have a background event C (see Exercise):

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)} .$$

Example: What is the probability that a product is good, if it was created by Machine I? We are given $P(A \mid B) = 0.58$, $P(A) = 0.57$ and $P(B) = 0.97$.

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{0.58 \cdot 0.97}{0.57} \approx 0.98 .$$

Independence

Two events A and B are **independent**, denoted by $A \perp B$, if

$$P(A \mid B) = P(A)$$

or, equivalently, iff

$$P(A \cap B) = P(A)P(B) .$$

If A and B are **independent**, learning that B happened *does not make A more or less likely to occur.*

Example: Suppose we roll a die. Let us consider the *events* A denoting *"the die outcome is even"* and B denoting *"the die outcome is either 1 or 2"*.

If the die is fair, then $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Moreover $A \cap B$ means the event that the outcome is two, so $P(A \cap B) = \frac{1}{6}$.

$$P(A \cap B) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(A)P(B) \quad \Rightarrow \quad A \text{ and } B \text{ are } \textit{independent}.$$

Conditional independence

Let A, B and C be *events*. A and B are **conditionally independent** given C, denoted by $A \perp\!\!\!\perp B \mid C$, iff

$$P(A \mid C) = P(A \mid B \cap C) ,$$

or, equivalently, iff

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C) .$$

A and B are **conditionally independent** given C means that *once we learned C, learning B gives us no additional information about A.*

Examples:

- The operation of a car's *starter motor* is conditionally independent its *radio* given the *status of the battery*.
- *Symptoms* are conditionally independent given the *disease*.

Summary *

- A **probability space** is a triple $(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A})$ is a *measurable space*, and P is a *measure* such that $P(\Omega) = 1$. If Ω is countable, then $(\Omega, \mathcal{A}, P)$ is called **discrete probability space**.
- Let $P(B) > 0$, then the **conditional probability of A given B** is defined as
$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)} .$$
- If A and B are **independent** ($A \perp B$), learning that B happened does not make A more or less likely to occur.
- A and B are **conditionally independent given** C, denoted by $A \perp\!\!\!\perp B \mid C$, means that once we learned C, learning B gives us no additional information about A.

In the **next lecture** we will learn about

- Random variables
- Probability distributions
- The Expectation-maximization algorithm

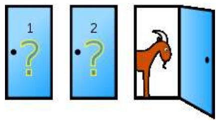
Literature *

1. Marek Capiński and Ekkerhard Kopp. *Measure, Integral and Probability*. Springer, 1998
2. Daphne Koller and Nir Friedman. *Probabilistic Graphical Models: Principles and Techniques*. MIT Press, 2009

A brain teaser *

Suppose you are on a game show and you are given the choice of three doors: Behind one door is a **car**; behind the others, **goats**.

You pick a door, say No. 1, and the *host*, who knows what is behind the doors, opens another door, say No. 3, which has a goat.



He then says to you, "Do you want to pick door No. 2?"

Question: Is it to your advantage to switch your choice?