# Probabilistic Graphical Models in Computer Vision (IN2329)

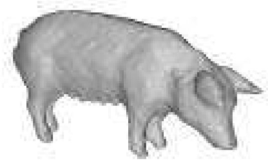
**Csaba Domokos**

Summer Semester 2015/2016

## Announcement: Computer Vision Group *

**Inquiries for Bachelor and Master projects are always welcome!**

We are currently working on the following `research topics`:



3D reconstruction

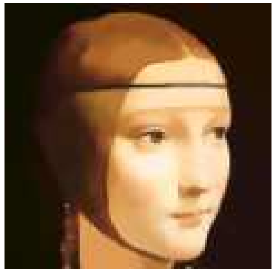
Optical flow

Shape analysis

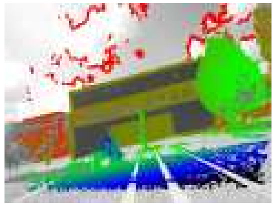
Robot vision

RGB-D vision

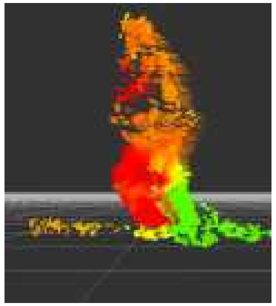
Image segmentation



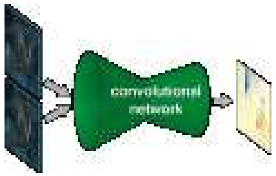
Convex relaxation



Visual SLAM



Scene flow



Deep learning

Please complete the form: `https://vision.in.tum.de/application`

**Agenda for today's lecture** *

1. Administration
2. Overview of the course
3. Introduction to Probability theory
   - Basic definitions
   - Conditional probability, Bayes' rule
   - Independence, conditional independence

### The course: IN2329 *

The course **Probabilistic Graphical Models in Computer Vision** will be organized as follows:

- Lectures: on Tuesdays at 10.00–12.00 in Room 00.13.036
- Tutorials: on Tuesdays at 14.00–16.00 in Room 02.05.014

The tutorials combines *theoretical* and *programming* assignments:

- **Assignment distribution**: Tuesday 11.00–11.15 in Room 00.13.036
- **Theoretical assignment due**: Tuesday 11.00–11.15 in Room 00.13.036
- **Assignment presentation**: Tuesday 14.00–16.00 in Room 02.05.014

Lecturer

Dr. Csaba Domokos (`csaba.domokos@in.tum.de`)

Teaching assistant (TA)

Lingni Ma (`lingni@in.tum.de`)

**Feel free to contact us!**

## April *

| April 2016 | | | | | | |
|---|---|---|---|---|---|---|
| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
| | | | | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12<br>Lecture 1<br>Tutorial 1 | 13 | 14 | 15 | 16 | 17 |
| 18 | 19<br>Lecture 2<br>Tutorial 2 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26<br>Lecture 3<br>Tutorial 3 | 27 | 28 | 29 | 30 | |

## May *

| May 2016 | | | | | | |
|---|---|---|---|---|---|---|
| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
| 2 | 3 Lecture 4 Tutorial 4 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 Lecture 5 Tutorial 5 | 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| 23 | 24 Lecture 6 Tutorial 6 | 25 | 26 | 27 | 28 | 29 |
| 30 | 31 Lecture 7 Tutorial 7 | | | | | |

**June** *

| June 2016 | | | | | | |
|---|---|---|---|---|---|---|
| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 Lecture 8 Tutorial 8 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 Lecture 9 Tutorial 9 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 Lecture 10 Tutorial 10 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 Lecture 11 Tutorial 11 | 29 | 30 | | | |

**July** *

| July 2016 | | | | | | |
|---|---|---|---|---|---|---|
| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
| | | | | 1 | 2 | 3 |
| 4 | 5 Lecture 12 Tutorial 12 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 Lecture 13 Tutorial 13 | 13 | 14 | 15 | 16 | 17 |
| 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 Week of the exam | 27 | 28 | 29 | 30 | 31 |

## Exam *

- The exam will be oral.
- According to our schedule, the exam will be held in the **last week of July, 25th–29th**.
- Students **need to be registered** prior to the exam: **May, 16th–June, 30th** via TUM online.

Participation at the tutorial:

- **Not mandatory, but highly recommended**:
  *Theoretical assignments* will help to understand the topics of the lecture. *Programming assignments* will help to apply the theory to practical Computer Vision problems.
- **Bonus**: Active students who solve *60% of the assignments* earn a bonus.
  If someone receives a mark between 1.3 and 4.0 in the *final exam*, the mark will be **improved by 0.3 and 0.4, respectively**.
  Note that marks of 1.0 and 5.0 cannot be improved!

## Bonus *

To achieve the bonus, the following requirements have to be fulfilled:

Theory

- **60%** of all theoretical assignments have to be solved.
  (Note that submissions happen **only** on Tuesdays at 11.00–11.15)
- The theoretical exercises have to *be presented in front of the class*.
  (The TA randomly selects a student who presents an exercise.)

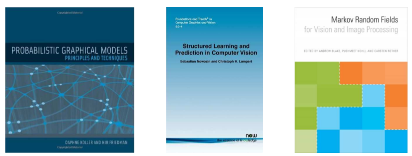
Programming

- **60%** of all programming assignments have to be presented during the tutorial.
- The programming exercises should *be explained to the TA*.

To promote team work, please form **groups of two or three students** in order to solve and submit the assignments.



## Recommended literature & prerequisites *




- D. Koller, N. Friedman. Probabilistic Graphical Models: Principles and Techniques, MIT Press, 2009.
- S. Nowozin, C. H. Lampert. Structured Learning and Prediction in Computer Vision, Foundations and Trends in Computer Graphics and Vision, 2011. Download
- A. Blake, P. Kohli, C. Rother. Markov Random Fields for Vision and Image Processing, MIT Press, 2011.

In addition, we will mention recent *conference* and *journal papers*.

**Prerequisites**: the course is intended for *Master students*.

- **Basic Mathematics**: multivariate analysis and linear algebra.
- **Basic Computer Science**: dynamic programming and basic data structures.

## Course Page *

On the **internal site** of the `course page` you could access to extra course material:
`https://vision.in.tum.de/teaching/ss2016/lecture_graphical_models/material`

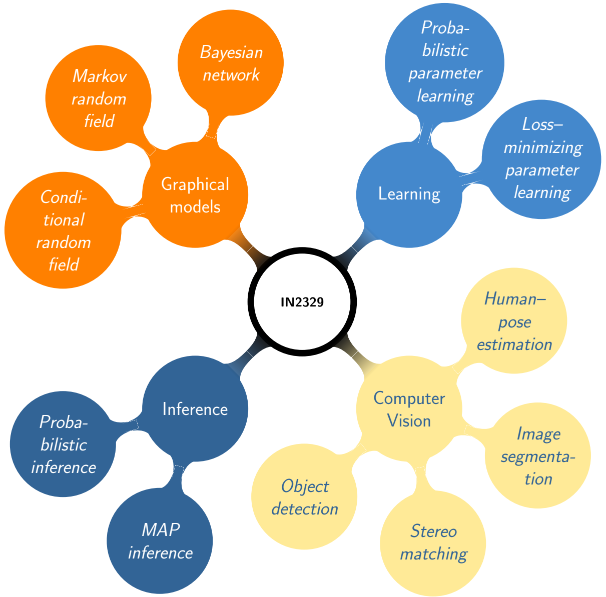
Password: `PGMCV:SS16`

Course materials:

- Printer friendly **slides for each lecture** (available prior to the lecture)
- **Assignment sheets** (available after the lecture)
- **Solution sheets** (available after the tutorial)

The course page will also be used for extra announcements.

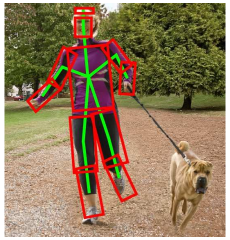
**Overview of the lecture** *



## Human–pose estimation *

The goal is to recognize an *articulated object* (i.e. human body) with different connecting parts (e.g., head, torso, left arm, . . . ).


Input image


Human pose estimation

An object is composed of a number of *rigid parts*, where each part is modeled as a **rectangle**. The *connections* encode generic relationships such as "close to", "to the left of".



## Binary image segmentation *

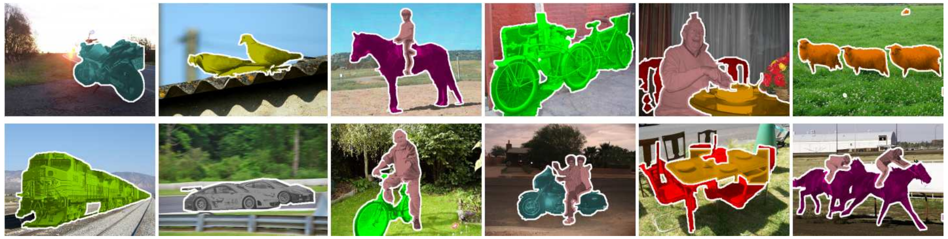
The goal is to give a binary label $y_i \in \mathbb{B} \triangleq \{0, 1\}$ for each pixel $i$, where 0 and 1 mean the *background* (a.k.a. ground) and the *foreground* (a.k.a. figure), respectively.


Input image


Figure–ground segmentation

## Semantic image segmentation *

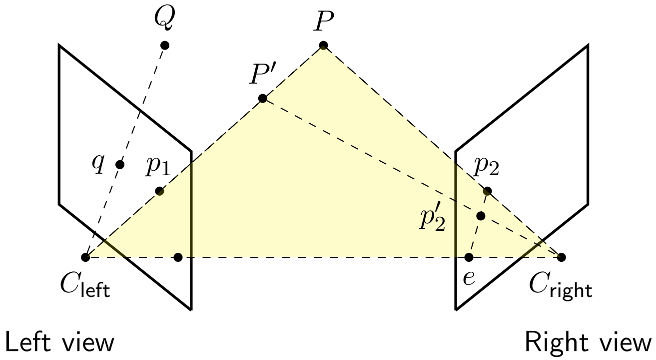
The goal is to give a label $y_i \in \mathcal{L} = \{1, 2, \ldots, c\}$ for each pixel $i$ according to its *semantic meanings*.



Exemplar semantic segmentations



## Stereo matching *




Given two images (i.e. left and right), an observed 2D point $p_1$ on the *left image*, corresponding to a 3D point $P$ that is situated on a line in $\mathbb{R}^3$. This line will be observed as a line on the *right image*.

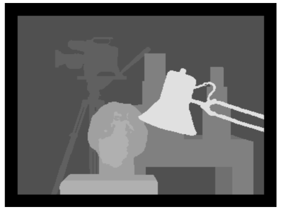
The pixels $p_1$ and $p_2$ corresponding to $P$ should should have similar intensities.

## Stereo matching *

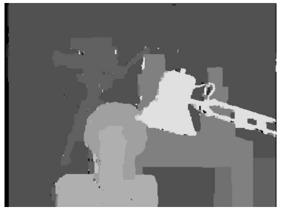
Usually we assume **rectified images** (i.e. the directions of the cameras are parallel).

Left view

Right view



Ground truth (depth map)



Result (depth map)

## Object detection *

We address the problem of *binary image segmentation*, where we also assume non-local **parameters** that are **known a priori**. For example, one can assume prior knowledge about the **shape** of the foreground.

Exemplar binary segmentation of cars assuming shape prior

**You may realize that we will mainly deal with labelling problems.**

### Reasoning under uncertainty

We often want to understand a system when we have *imperfect* or *incomplete* information due to, for example, noisy measurement.
There are *two main reasons* why we might **reason under uncertainty**:

- *Laziness*: modeling every detail of a complex system is *costly*.
- *Ignorance*: we *may not completely understand*.

**Probability** $P(A)$ refers to a degree of confidence that an event $A$ with uncertain nature will occur.

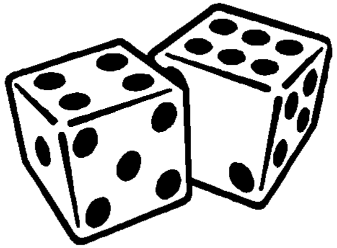
It is common to assume that $0 \leqslant P(A) \leqslant 1$:

- If $P(A) = 1$, we are certain that $A$ occurs,
- while $P(A) = 0$ asserts that $A$ will not occur.

## Experiment, event space, event

An **experiment** is a (random) process that can be infinitely many times repeated and has a well-defined set of possible **outcomes**. In case of repeated experiments the individual repetitions are also called **trials**.

*Example*: throwing two "fair dice" (i.e. we assume equally likely chance of landing on any face) with six faces.



The **event space**, denoted by $\Omega$, is the set of possible outcomes.

*Example*: $\Omega = \{(i, j) : 1 \leqslant i, j \leqslant 6\}$.

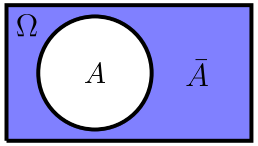
A set of outcomes $A \subseteq \Omega$ is called an **event**. An **atomic event** is an event that contains a single outcome $\omega \in \Omega$.

*Example*: $A = \{(i, j) : i + j = 11\}$, i.e. the sum of the numbers showing on the top is equal to eleven.

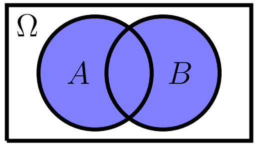
## Basic notations

Let $A$ and $B$ be two *events* from an *event space* $\Omega$. We will use the following notations:

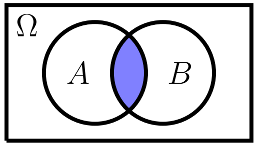
$A$ does not occur: $\bar{A} = \Omega \backslash A$



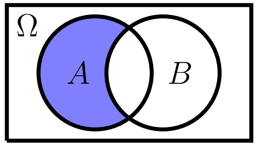

either $A$ or $B$ occur: $A \cup B$




both $A$ and $B$ occur: $A \cap B$




$A$ occurs and $B$ does not: $A \backslash B$




- The $\emptyset$ is called the **impossible event**; and
- $\Omega$ is the **sure event**.

## Discrete probability space

A *probability space* represents our **uncertainty** regarding an *experiment*.

A triple $(\Omega, \mathcal{A}, P)$ is called a **discrete probability space**, if

- $\Omega$ is not empty and **countable** (i.e. $\exists \mathcal{S} \subseteq \mathbb{N}$ such that $|\Omega| = |\mathcal{S}|$),
- $\mathcal{A}$ is the **power set** $\mathcal{P}(\Omega)$ (i.e. the set of all subsets of $\Omega$), and
- $P : \mathcal{A} \to \mathbb{R}$ is a function, called a **probability measure**, with the following properties:

  1. $P(A) \geqslant 0$ for all $A \in \mathcal{A}$
  2. $P(\Omega) = 1$
  3. **$\sigma$-additivity** holds: if $A_n \in \mathcal{A}, n = 1, 2, \ldots$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) \; .$$

The conditions 1-3. are called **Kolmogorov's axioms**.

## Example: throwing two "fair dice" *

For this case a *discrete probability space* $(\Omega, \mathcal{A}, P)$ is given by

- **Event space**: $\Omega = \{(i,j) : 1 \leqslant i,j \leqslant 6\}$.
- $\mathcal{A} = \mathcal{P}(\Omega) = \{\{(1,1)\}, \ldots, \{(1,1),(1,2)\}, \ldots, \{(1,1),(1,2),(1,3)\}, \ldots\}$.
- The **probability measure**

$$P(A) = \frac{|A|}{36} = \frac{k}{36} ,$$

where $k$ is the number of *atomic events* in $A$.

*Example*: Let $A$ denote the *event* that *"the sum of the numbers showing on the top is equal to eleven"*, that is

$$A = \{(i,j) : i + j = 11\} = \{(5,6),(6,5)\} .$$

Hence

$$P(A) = P(\{(5,6),(6,5)\}) = \frac{2}{36} .$$



## $\sigma$-algebra, measure, measure space

Assume an arbitrary set $\Omega$ and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. The set $\mathcal{A}$ is a **$\sigma$-algebra over $\Omega$** if the following conditions are satisfied:

1. $\varnothing \in \mathcal{A}$,
2. $A \in \mathcal{A} \Rightarrow \bar{A} \in \mathcal{A}$ (i.e. it is *closed under complementation*),
3. $A_i \in \mathcal{A}\ (i \in \mathbb{N}) \Rightarrow \bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$ (i.e. it is *closed under countable union*).

It is a consequence of this definition that $\Omega \in \mathcal{A}$ is also satisfied. (See exercise.)
Assume an *arbitrary set* $\Omega$ and a *$\sigma$-algebra* $\mathcal{A}$ over $\Omega$. A function $P : \mathcal{A} \to [0, \infty]$ is called a **measure** if the following conditions are satisfied:

1. $P(\varnothing) = 0$,
2. $P$ is $\sigma$-additive.

Let $\mathcal{A}$ be a *$\sigma$-algebra* over $\Omega$ and $P : \mathcal{A} \to [0, \infty]$ is a *measure*. $(\Omega, \mathcal{A})$ is said to be a **measurable space** and the triple $(\Omega, \mathcal{A}, P)$ is called a **measure space**.

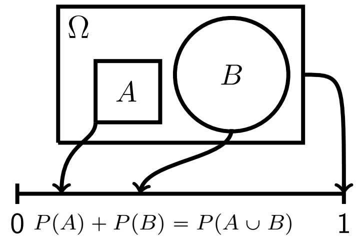
## Probability space

A **probability space** is a triple $(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A})$ is a *measurable space*, and $P$ is a *measure* such that $P(\Omega) = 1$, called a **probability measure**.

*To summarize*:
A triple $(\Omega, \mathcal{A}, P)$ is called **probability space**, if

- the **event space** $\Omega$ is *not empty*,
- $\mathcal{A}$ is a $\sigma$-**algebra** over $\Omega$, and
- $P : \mathcal{A} \to \mathbb{R}$ is a function with the following properties:
  1. $P(A) \geqslant 0$ for all $A \in \mathcal{A}$
  2. $P(\Omega) = 1$
  3. $\sigma$-**additive**: if $A_n \in \mathcal{A}$, $n = 1, 2, \ldots$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n) \ .$$



### Example: throwing a dart *

Suppose a dart is thrown at a round board modeled as a unit circle. The **event space** contains the location of the dart if it lands in the board only. Hence it is given by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\} .$$

We denote the **area** of an the **event** $A \subseteq \Omega$ by $\mu(A)$, which is defined as the *Riemann-integral* of the **characteristic function of** $A$

$$\mu(A) := \int_\Omega \chi_A(x)\mathrm{d}x , \quad \text{where} \quad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A . \end{cases}$$

The $\sigma$-**algebra** $\mathcal{A}$ over $\Omega$ is defined as follows

$$\mathcal{A} = \{A \subseteq \Omega : \mu(A) \text{ exists}\} .$$

The **probability measure** $P : \Omega \to [0, 1]$ is given by $P(A) = \frac{\mu(A)}{\pi}$ .

## Some simple consequences of the axioms

The following rules are frequently used in applications:

- $P(A) = 1 - P(\Omega \backslash A)$.

  *Proof.* Note that $A$ and $\Omega \backslash A$ are disjoint.
  $1 = P(\Omega) = P(A \cup (\Omega \backslash A)) = P(A) + P(\Omega \backslash A)$. □

- $P(\varnothing) = 0$.

  *Proof.* $P(\varnothing) = 1 - P(\Omega \backslash \varnothing) = 1 - P(\Omega) = 1 - 1 = 0$. □

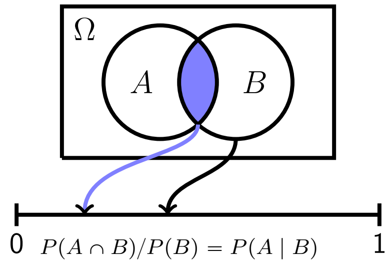
- If $A \subseteq B$, then $P(A) \leqslant P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B) \leqslant P(A) + P(B)$.
- $P(A \backslash B) = P(A) - P(A \cap B)$.
- $\ldots$

### Conditional probability

Conditional probability allows us to reason with *partial information*.
If $P(B) > 0$, the **conditional probability of $A$ given $B$** is defined as

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)} \,.$$

This is the probability that $A$ occurs, given we have observed $B$, i.e. we know the experiment's actual outcome will be in $B$.




Note that the *axioms and rules of probability theory are fulfilled* for the conditional probability. (e.g., $P(A \mid B) = 1 - P(\bar{A} \mid B)$).

**Example** *

Consider two producing machines creating identical product in a factory. Assume we are given the following table with probabilities

| | Machine I | Machine II | |
|---|---|---|---|
| The product is good | 0.56 | 0.41 | 0.97 |
| The product is waste | 0.01 | 0.02 | 0.03 |
| | 0.57 | 0.43 | 1 |

*Question*: What is the probability of a product was created by Machine I, when it is good?

Let $A$ denote the event that *"the product was created by Machine I"* and let $B$ denote the event that *"the product is good"*.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.56}{0.97} \approx 0.58 \ .$$



**The chain rule**

Starting with the definition of *conditional probability* $P(B \mid A)$ and multiplying by P(A) we get the **product rule**:

$$P(A \cap B) = P(A)P(B \mid A) \ .$$

**The chain rule** is given by

$$P(\cap_{i=1}^{n} A_i) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid \cap_{i=1}^{n-1} A_i) \ . \tag{1}$$

*Proof.* By induction. For $n = 2$ we get the product rule. Let $n \in \mathbb{N}$ be given and suppose Eq. (1) is true for $k \leqslant n$. Then

$$P(\cap_{i=1}^{n+1} A_i) = P(A_{n+1} \cap (\cap_{i=1}^{n} A_i)) = P(A_{n+1} \mid \cap_{i=1}^{n} A_i)P(\cap_{i=1}^{n} A_i) \ .$$

□

The chain rule will become important later when we discuss *conditional independence*.

## Bayes' rule

By making use of the product rule we can get

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B)} .$$

$P(A \mid B)$ is often called the **posteriori probability**, and $P(B \mid A)$ is called the **likelihood**, and $P(A)$ is called the **prior probability**.

A more general version of **Bayes' rule**, when we have a background event $C$:

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)} .$$

*Example*: What is the probability that a product is good, if it was created by Machine I? We are given $P(A \mid B) = 0.58$, $P(A) = 0.57$ and $P(B) = 0.97$.

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{0.58 \cdot 0.97}{0.57} \approx 0.98 .$$



## Independence

Two events $A$ and $B$ are **independent**, denoted by $A \perp B$, if

$$P(A \mid B) = P(A)$$

or, equivalently, iff

$$P(A \cap B) = P(A)P(B) .$$

If $A$ and $B$ are **independent**, learning that $B$ happened *does not make $A$ more or less likely to occur.*

*Example*: Suppose we roll a die. Let us consider the *events* $A$ denoting *"the die outcome is even"* and $B$ denoting *"the die outcome is either 1 or 2"*.

If the die is fair, then $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Moreover $A \cap B$ means the event that the outcome is two, so $P(A \cap B) = \frac{1}{6}$.

$$P(A \cap B) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(A)P(B) \quad \Rightarrow \quad A \text{ and } B \text{ are } \textit{independent}.$$

## Conditional independence

Let $A$, $B$ and $C$ be *events*. $A$ and $B$ are **conditionally independent** given $C$, denoted by $A \perp\!\!\!\perp B \mid C$, iff

$$P(A \mid C) = P(A \mid B \cap C) \; ,$$

or, equivalently, iff

$$P(A \cap B \mid C) = P(A \mid C) P(B \mid C) \; .$$

$A$ and $B$ are **conditionally independent** given $C$ means that *once we learned $C$, learning $B$ gives us no additional information about $A$.*

*Examples:*

- The operation of a car's *starter motor* is conditionally independent its *radio* given the *status of the battery*.
- *Symptoms* are conditionally independent given the *disease*.



## Summary *

- A **probability space** is a triple $(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A})$ is a *measurable space*, and $P$ is a *measure* such that $P(\Omega) = 1$.
- Let $P(B) > 0$, then the **conditional probability of $A$ given $B$** is defined as

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)} \; .$$

- If $A$ and $B$ are **independent** ($A \perp B$), learning that $B$ happened does not make $A$ more or less likely to occur.
- $A$ and $B$ are **conditionally independent given** $C$, denoted by $A \perp\!\!\!\perp B \mid C$, means that once we learned $C$, learning $B$ gives us no additional information about $A$.

In the **next lecture** we will learn about

- *Random variables*
- *Probability distributions*
- *Expectation-maximization algorithm*

## Literature *

[1] Marek Capiński and Ekkerhard Kopp. *Measure, Integral and Probability*. Springer, 1998.

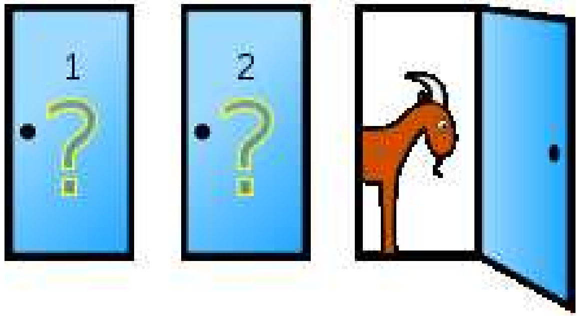
[2] Daphne Koller and Nir Friedman. *Probabilistic Graphical Models: Principles and Techniques*. MIT Press, 2009.



## A brain teaser *

Suppose you are on a game show and you are given the choice of three doors: Behind one door is a **car**; behind the others, **goats**.

You pick a door, say No. 1, and the *host*, who knows what is behind the doors, opens another door, say No. 3, which has a goat.




He then says to you, "Do you want to pick door No. 2?"

*Question*: Is it to your advantage to switch your choice?