Probabilistic Graphical Models in Computer Vision (IN2329)

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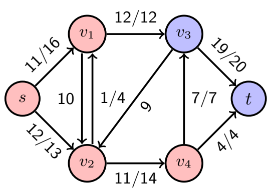
Summer Semester 2015/2016

5. Move making algorithms

Agenda for today's lecture *

In the **previous lecture** we learnt about the minimum $s-t$ cut problem



Today we are going to learn about

- Exact solution for *binary image segmentation* via *graph cut*
- Approximate solutions for the multi-label problem:
  - α-expansion
  - $\alpha-\beta$ swap

Binary image segmentation

Regular functions *

Let us consider a function f of two binary variables, then f is called **regular**, if it satisfies the following inequality

$$f(0,0)+f(1,1) \leqslant f(0,1)+f(1,0) .$$

Example: the **Potts-model** is *regular*, i.e.

$$[\![0 \neq 0]\!]+[\![1 \neq 1]\!]=0 \leqslant 2=[\![0 \neq 1]\!]+[\![1 \neq 0]\!] .$$

Regular energy functions

Let us consider an *energy function* E of n binary variables which can be written as the sum of functions of up to two variables, that is

$$E(y_1,\dots,y_n)=\sum_i E_i(y_i)+\sum_{i<j} E_{ij}(y_i,y_j) .$$

E is *regular*, if each term E_{ij} for all $i<j$ satisfies

$$E_{ij}(0,0)+E_{ij}(1,1) \leqslant E_{ij}(0,1)+E_{ij}(1,0) .$$

If each term E_{ij} is *regular*, then it is possible to find the **global** minimum of E in *polynomial time* by solving a *minimum $s-t$ cut problem*.

Binary image segmentation

We have already seen that **binary image segmentation** can be reformulated as the minimization of an *energy function* $E:\{0,1\}^{\mathcal{V}}\times\mathcal{X}\to\mathbb{R}$:

$$E(\mathbf{y};\mathbf{x})=\sum_{i\in\mathcal{V}} E_i(y_i;x_i)+\sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i,y_j;x_i,x_j) .$$

where $\mathcal{V}$ corresponds to the output variables, i.e. the pixels, and $\mathcal{E}$ includes the pairs of neighboring pixels.

Assume probability densities p_b and p_f estimated for the background and the foreground, respectively. This can be achieved, for example, by making use of the `EM algorithm`. The **unary energies** E_i for all $i\in\mathcal{V}$ can be defined as

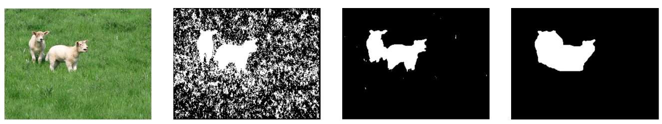
$$E_i(0,x_i)=0$$
$$E_i(1,x_i)=\log\frac{p_b(x_i)}{p_f(x_i)} .$$

Contrast-sensitive Potts-model

The pairwise energy functions are defined as

$$E_{ij}(y_i,y_j;x_i,x_j)=w\exp(-\gamma\|x_i-x_j\|^2)[\![y_i \neq y_j]\!] ,$$

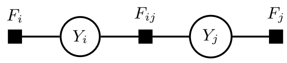
where $w \geqslant 0$ is a weighting factor. The parameter γ is the mean edge strength.



Original image | w is small | w is medium | w is high

Energy minimization via minimum $s-t$ cut

Let us consider the following example



Through this example we illustrate how to minimize *regular energy functions* consisting of up to pairwise relationships. In our example $\mathbf{y} \in \{0,1\}^2$ and $E(\mathbf{y})$ is defined as

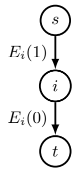
$$E(\mathbf{y}) = E_1(y_1) + E_2(y_2) + E_{12}(y_1, y_2) .$$

We will create a flow network $(\mathcal{V} \cup \{s,t\}, \mathcal{E}', c, s, t)$ such that the minimum $s-t$ cut will correspond to the minimization of our energy function $E(\mathbf{y})$, where the labeling for each $i \in \mathcal{V}$ is given by

$$y_i = \begin{cases} 0, & \text{if } i \in \mathcal{S} , \\ 1, & \text{if } i \in \mathcal{T} . \end{cases}$$

Graph construction: unary energies

Let us consider the unary energy function $E_i : \{0,1\} \to \mathbb{R}$.

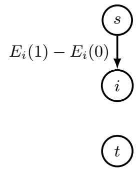


Obviously, the minimum $s-t$ cut of the flow network will correspond to

$$\operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) .$$

Without loss of generality we can assume that $E_i(1) > E_i(0)$, then we can write

$$\operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) = \operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) - E_i(0) .$$



Graph construction: pairwise energies

Let us consider the pairwise energy function $E_{ij}(y_i, y_j) : \{0,1\}^2 \to \mathbb{R}$. The possible values of $E_{ij}(y_i, y_j)$ are shown in the table:

E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

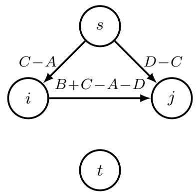
We furthermore assume that $E_{ij}(y_i, y_j)$ is regular, that is

$$E_{ij}(0,0) + E_{ij}(1,1) \leqslant E_{ij}(0,1) + E_{ij}(1,0)$$
$$A + D \leqslant B + C .$$

Let us note that $E_{ij}(y_i, y_j)$ can be decomposed as:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = A + \begin{bmatrix} 0 & B-A \\ C-A & D-A \end{bmatrix} = A + \begin{bmatrix} 0 & 0 \\ C-A & C-A \end{bmatrix} + \begin{bmatrix} 0 & B-A \\ 0 & D-C \end{bmatrix}$$
$$= A + \underbrace{\begin{bmatrix} 0 & 0 \\ C-A & C-A \end{bmatrix}}_{E_i(1)} + \underbrace{\begin{bmatrix} 0 & D-C \\ 0 & D-C \end{bmatrix}}_{E_j(1)} + \underbrace{\begin{bmatrix} 0 & B+C-A-D \\ 0 & 0 \end{bmatrix}}_{B+C-A-D \geqslant 0}$$

Graph construction: pairwise energies, $C-A \geqslant 0$, $D-C \geqslant 0$ *

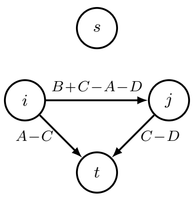


E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 0$.

$$C - A \geqslant 0 \;\Rightarrow C \geqslant A .$$
$$D - C \geqslant 0 \;\Rightarrow D \geqslant C \;\Rightarrow\; D \geqslant A .$$
$$0 \leqslant B + C - A - D \leqslant B - A \;\Rightarrow B \geqslant A .$$

Graph construction: pairwise energies, $A-C \geqslant 0$, $C-D \geqslant 0$ *

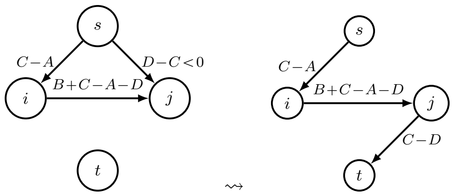


E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 1$.

$$C - D \geqslant 0 \;\Rightarrow C \geqslant D .$$
$$A - C \geqslant 0 \;\Rightarrow A \geqslant C \;\Rightarrow\; A \geqslant D .$$
$$0 \leqslant B + C - A - D \leqslant B - D \;\Rightarrow B \geqslant D .$$

Graph construction: pairwise energies, $C-A \geqslant 0$, $C-D \geqslant 0$ *

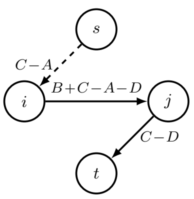


Note that the labeling $y_i = 1$, $y_j = 0$ in this case is not possible, since

$$C - A \geqslant 0 \;\Rightarrow\; C \geqslant A .$$

Graph construction: pairwise energies, $C-A \geqslant 0$, $C-D \geqslant 0$ *

Assume that $\min\{C-A, B+C-A-D, C-D\} = C-A$.



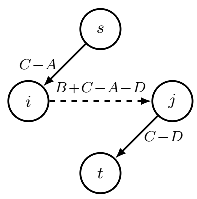
E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 1$.

$$C - A \leqslant B + C - A - D \;\Rightarrow 0 \leqslant B - C \;\Rightarrow\; B \geqslant D .$$
$$C - A \geqslant C - D \;\Rightarrow A \geqslant D .$$
$$C - D \geqslant 0 \;\Rightarrow C \geqslant D .$$

Graph construction: pairwise energies, $C-A \geqslant 0$, $C-D \geqslant 0$ *

Assume that $\min\{C-A, B+C-A-D, C-D\} = B+C-A-D$.



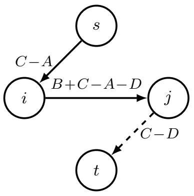
E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = 0$, $y_j = 1$.

$$B + C - A - D \leqslant C - A \;\Rightarrow B \leqslant D .$$
$$B + C - A - D \geqslant C - D \;\Rightarrow B \leqslant A .$$
$$C - A \geqslant 0 \;\Rightarrow A \leqslant C \;\Rightarrow\; B \leqslant C .$$

Graph construction: pairwise energies, $C-A\geqslant 0$, $C-D\geqslant 0$ *

Assume that $\min\{C-A, B+C-A-D, C-D\} = C-D$.

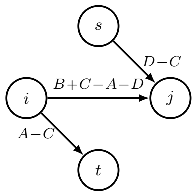


E_{ij}	$y_j=0$	$y_j=1$
$y_i=0$	A	B
$y_i=1$	C	D

Labeling: $y_i = y_j = 0$.

$$C-D \leqslant B+C-A-D \;\Rightarrow B \geqslant A\,.$$
$$C-D \leqslant C-A \;\Rightarrow D \geqslant A\,.$$
$$C-D \geqslant 0 \;\Rightarrow C \geqslant D \;\Rightarrow\; C \geqslant A\,.$$

Graph construction: pairwise energies, $A-C\geqslant 0$, $D-C\geqslant 0$ *



E_{ij}	$y_j=0$	$y_j=1$
$y_i=0$	A	B
$y_i=1$	C	D

Labeling: $y_i = 1$, $y_j = 0$.

$$D-C \geqslant 0 \;\Rightarrow D \geqslant C\,.$$
$$A-C \geqslant 0 \;\Rightarrow A \geqslant C\,.$$
$$0 \leqslant B+C-A-D \leqslant B-A \;\Rightarrow B \geqslant A \;\Rightarrow\; B \geqslant C\,.$$

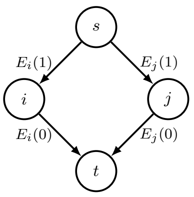
All the other cases can be similarly derived.

Graph construction

Putting all together we get that

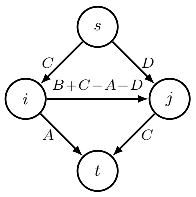
Unaries

$$\operatorname*{argmin}_{\mathbf{y}} E_i(y_i)+E_j(y_j)$$



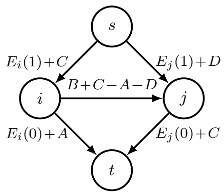
Pairwise

$$\operatorname*{argmin}_{\mathbf{y}} E_{ij}(y_i,y_j)$$



Overall energy

$$\operatorname*{argmin}_{\mathbf{y}} E_i(y_i)+E_j(y_j) + E_{ij}(y_i,y_j)$$



Remarks

Regularity is an *extremely important* property as is allows to minimize energy functions by making use of graph cut. Moreover, without the regularity constraint, the problem become intractable.

Let E_2 be a nonregular function of two binary variables. Then minimizing the energy function

$$E(y_1,\ldots,y_n) = \sum_i E_i(y_i) + \sum_{i<j} E_2(y_i,y_j)\,,$$

where E_i are arbitrary functions of one binary variable, is NP-hard.

Multi-label problem

Multi-label problem

We define a label set $\mathcal{L} = \{1,2,\ldots,L\}$, where L is a (finite) constant. Therefore the output domain is defined as $\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$. The *energy function* has the following form

$$E(\mathbf{y};\mathbf{x}) = \sum_{i\in\mathcal{V}} E_i(y_i;\mathbf{x}) + \sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i,y_j;\mathbf{x})\,,$$

where $\mathbf{x}$ consists of an input image.

In order to ease to notation we will omit $\mathbf{x}$ and define the *energy function* simply as

$$E(\mathbf{y}) = \sum_{i\in\mathcal{V}} E_i(y_i) + \sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i,y_j)\,.$$

Metric *

A function $d:\mathcal{L}\times\mathcal{L}\to\mathbb{R}^+$ is called a **metric** if the following properties are satisfied:

1. **Identity of indiscernibles**: $d(\ell_1,\ell_2)=0 \quad\Leftrightarrow\quad \ell_1=\ell_2$ for all $\ell_1,\ell_2\in\mathcal{L}$.
2. **Symmetry**: $d(\ell_1,\ell_2)=d(\ell_2,\ell_1)$ for all $\ell_1,\ell_2\in\mathcal{L}$.
3. **Triangle inequality**: $d(\ell_1,\ell_3)\leqslant d(\ell_1,\ell_2)+d(\ell_2,\ell_3)$ for all $\ell_1,\ell_2,\ell_3\in\mathcal{L}$.

Example: the **truncated absolute distance** $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $d(x,y)=\min(K,|x-y|)$ is a *metric*, where K is some constant. (See Exercise)

If a function $d:\mathcal{L}\times\mathcal{L}\to\mathbb{R}$ satisfies the first two properties (i.e. identity of indiscernibles and symmetric), then it is called **semi-metric**.

Example: the **truncated quadratic function** $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $d(x,y)=\min(K,|x-y|^2)$ is a *semi-metric*, where K is some constant. (See Exercise)

$\alpha-\beta$ swap

$\alpha - \beta$ swap

$\alpha - \beta$ swap changes the variables that are labeled as $\ell \in \{\alpha, \beta\}$. Each of these variables can choose either α or β. We introduce the following notation

$$\mathcal{Z}_{\alpha\beta}(\mathbf{y}, \alpha, \beta) = \{\mathbf{z} \in \mathcal{Y} : z_i = y_i, \text{ if } y_i \notin \{\alpha, \beta\}, \text{ otherwise } z_i \in \{\alpha, \beta\}\} .$$

The minimization of the *energy function* E can be reformulated as follows:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} E(\mathbf{z}) = \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{i \in \mathcal{V}} E_i(z_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(z_i, z_j)$$

$$= \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \Big[\underbrace{\sum_{i \in \mathcal{V},\, y_i \notin \{\alpha,\beta\}} E_i(y_i)}_{\text{constant}} + \underbrace{\sum_{i \in \mathcal{V},\, y_i \in \{\alpha,\beta\}} E_i(z_i)}_{\text{unary}}$$

$$+ \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \notin \{\alpha,\beta\}}} E_{ij}(y_i, y_j)}_{\text{constant}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \in \{\alpha,\beta\}, y_j \notin \{\alpha,\beta\}}} E_{ij}(z_i, y_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \notin \{\alpha,\beta\}, y_j \in \{\alpha,\beta\}}} E_{ij}(y_i, z_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \in \{\alpha,\beta\}}} E_{ij}(z_i, z_j)}_{\text{pairwise}} \Big] .$$

Local optimization

Let us consider $E_{ij}(z_i, z_j)$ for a given $(i, j) \in \mathcal{E}$:

E_{ij}	α	β
α	$E_{ij}(\alpha, \alpha)$	$E_{ij}(\alpha, \beta)$
β	$E_{ij}(\beta, \alpha)$	$E_{ij}(\beta, \beta)$

If we assume that $E_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is a **semi-metric** for each $(i, j) \in \mathcal{E}$, then

$$E_{ij}(\alpha, \alpha) + E_{ij}(\beta, \beta) = 0 \leqslant E_{ij}(\alpha, \beta) + E_{ij}(\beta, \alpha) = 2E_{ij}(\alpha, \beta) ,$$

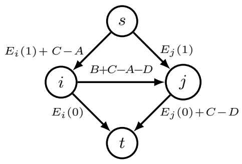
which means that E_{ij} is **regular** w.r.t. the labeling $\mathcal{Z}_{\alpha\beta}(\mathbf{y}, \alpha, \beta)$.

Graph construction for semi-metrics

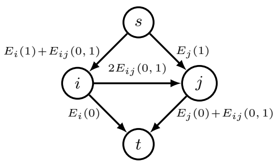
Let us consider the following *binary energy function*:

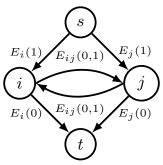
$$E(\mathbf{y}) = E_i(y_i) + E_j(y_j) + E_{ij}(y_i, y_j) ,$$

where E_{ij} is assumed to be a *semi-metric*.



Since E_{ij} is a *semi-metric*, we can construct a flow for $E(\mathbf{y})$ as follows:





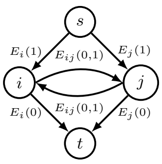
y_i	y_j	$E(\mathbf{y})$
0	0	$E_i(0)+E_j(0)$
0	1	$E_i(0)+E_j(1)+E_{ij}(1,0)$
1	0	$E_i(1)+E_j(0)+E_{ij}(1,0)$
1	1	$E_i(1)+E_j(1)$

Graph construction: t-links

We need to minimize the following **regular** *energy function*:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{\substack{i \in \mathcal{V} \\ y_i \in \{\alpha,\beta\}}} E_i(z_i) + \sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \in \{\alpha,\beta\}, y_j \notin \{\alpha,\beta\}}} E_{ij}(z_i, y_j) + \sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \notin \{\alpha,\beta\}, y_j \in \{\alpha,\beta\}}} E_{ij}(y_i, z_j) + \sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \in \{\alpha,\beta\}}} E_{ij}(z_i, z_j) .$$

Based on construction applied for *binary image segmentation*, we can also define a *flow network* $(\mathcal{V}', \mathcal{E}', c, \alpha, \beta)$, where $\mathcal{V}' = \{\alpha, \beta\} \cup \{i \in \mathcal{V} : y_i \in \{\alpha, \beta\}\}$ and $\mathcal{E}' = \underbrace{\{(\alpha, i), (i, \beta) : i \in \mathcal{V}' \backslash \{\alpha, \beta\}\}}_{\text{t-links}} \cup \underbrace{\{(i, j), (j, i) \mid i, j \in \mathcal{V}' \backslash \{\alpha, \beta\}, (i, j) \in \mathcal{E}\}}_{\text{n-links}}$.

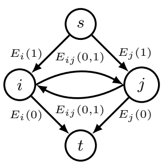


t-links: for all $i \in \mathcal{V}' \backslash \{\alpha, \beta\}$

$$c(\alpha, i) = E_i(\beta) + \sum_{(i,j) \in \mathcal{E},\, y_j \notin \{\alpha,\beta\}} E_{ij}(\beta, y_j) + \sum_{(j,i) \in \mathcal{E},\, y_j \notin \{\alpha,\beta\}} E_{ji}(y_j, \beta) .$$

$$c(i, \beta) = E_i(\alpha) + \sum_{(i,j) \in \mathcal{E},\, y_j \notin \{\alpha,\beta\}} E_{ij}(\alpha, y_j) + \sum_{(j,i) \in \mathcal{E},\, y_j \notin \{\alpha,\beta\}} E_{ji}(y_j, \alpha) .$$

Graph construction: n-links



n-links: for all $(i, j) \in \mathcal{E}$, where $i, j \in \mathcal{V}' \backslash \{\alpha, \beta\}$

$$c(i, j) = c(j, i) = E_{ij}(\alpha, \beta) .$$

$\alpha - \beta$ swap algorithm *

Input: An *energy function* $E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$ to be minimized, where E_{ij} is a **semi-metric** for each $(i, j) \in \mathcal{E}$

Output: A local minimum $\mathbf{y} \in \mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ of $E(\mathbf{y})$

1: Choose an arbitrary initial labeling $\mathbf{y} \in \mathcal{Y}$
2: $\hat{\mathbf{y}} \leftarrow \mathbf{y}$
3: **for all** $(\alpha, \beta) \in \mathcal{L} \times \mathcal{L}$ **do**
4: find $\hat{\mathbf{z}} \in \operatorname{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\hat{\mathbf{y}},\alpha,\beta)} E(\mathbf{z})$
5: $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{z}}$
6: **end for**
7: **if** $E(\hat{\mathbf{y}}) < E(\mathbf{y})$ **then**
8: $\mathbf{y} \leftarrow \hat{\mathbf{y}}$
9: Goto Step 2
10: **end if**

$\alpha - \beta$ swap algorithm is guaranteed to terminate in a finite number of cycles. This algorithm computes at least $|\mathcal{L}|^2$ graph cuts, which may take a lot of time, even for moderately large label spaces.

α-expansion

α-expansion

α-expansion allows each variable either to keep its current label or to change it to the label $\alpha \in \mathcal{L}$. We introduce the following notation

$$\mathcal{Z}_{\alpha}(\mathbf{y}, \alpha) = \{\mathbf{z} \in \mathcal{Y} : z_i \in \{y_i, \alpha\} \text{ for all } i \in \mathcal{V}\} .$$

The minimization of the *energy function* E can be reformulated as follows:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha}(\mathbf{y},\alpha)} E(\mathbf{z}) = \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha}(\mathbf{y},\alpha)} \sum_{i \in \mathcal{V}} E_i(z_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(z_i, z_j)$$

$$= \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha}(\mathbf{y},\alpha)} \Big[\underbrace{\sum_{i \in \mathcal{V},\, y_i = \alpha} E_i(\alpha)}_{\text{constant}} + \underbrace{\sum_{i \in \mathcal{V},\, y_i \neq \alpha} E_i(z_i)}_{\text{unary}}$$

$$+ \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i = \alpha, y_j = \alpha}} E_{ij}(\alpha, \alpha)}_{\text{constant}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i = \alpha, y_j \neq \alpha}} E_{ij}(\alpha, z_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \neq \alpha, y_j = \alpha}} E_{ij}(z_i, \alpha)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \neq \alpha, y_j \neq \alpha}} E_{ij}(z_i, z_j)}_{\text{pairwise}} \Big] .$$

Local optimization

Let us consider $E_{ij}(z_i, z_j)$ for a given $(i,j) \in \mathcal{E}$:

E_{ij}	α	y_j
α	$E_{ij}(\alpha,\alpha)$	$E_{ij}(\alpha, y_j)$
y_i	$E_{ij}(y_i,\alpha)$	$E_{ij}(y_i, y_j)$

If we assume that $E_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is a **metric** for each $(i,j) \in \mathcal{E}$, then

$$E_{ij}(\alpha,\alpha) + E_{ij}(y_i,y_j) = E_{ij}(y_i,y_j) \leqslant E_{ij}(y_i,\alpha) + E_{ij}(\alpha,y_j) ,$$

which means that E_{ij} is **regular** w.r.t. the labeling $\mathcal{Z}_\alpha(\mathbf{y},\alpha)$.

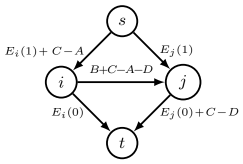
Graph construction

We need to minimize the following **regular** *energy function*:

$$\hat{\mathbf{z}} \in \underset{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)}{\operatorname{argmin}} \underbrace{\sum_{\substack{i\in\mathcal{V}\\ y_i\neq\alpha}} E_i(z_i) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i=\alpha,\, y_j\neq\alpha}} E_{ij}(\alpha, z_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j=\alpha}} E_{ij}(z_i,\alpha)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j\neq\alpha}} E_{ij}(z_i,z_j)}_{\text{pairwise}} .$$

Based on construction applied for *binary image segmentation*, we can also define a *flow network* $(\mathcal{V}', \mathcal{E}', c, \alpha, \bar{\alpha})$, where $\mathcal{V}' = \{\alpha,\bar{\alpha}\} \cup \{i \in \mathcal{V} : y_i \neq \alpha\}$ and $\mathcal{E}' = \underbrace{\{(\alpha,i),(i,\bar{\alpha}) : i \in \mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}\}}_{\text{t-links}} \cup \underbrace{\{(i,j)\in\mathcal{E} : i,j \in \mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}\}}_{\text{n-links}}$.

Graph construction: t-links

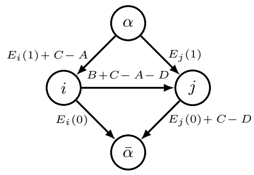


t-links: for all $i \in \mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}$

$$c(\alpha,i) = E_i(y_i) + \sum_{(i,j)\in\mathcal{E},\, y_j=\alpha} E_{ij}(y_i,\alpha) + \sum_{(j,i)\in\mathcal{E},\, y_j=\alpha} E_{ji}(\alpha,y_i) + \underbrace{\sum_{(i,j)\in\mathcal{E},\, y_j\neq\alpha} E_{ij}(y_i,\alpha)}_{C} .$$

$$c(i,\bar{\alpha}) = E_i(\alpha) + \underbrace{\sum_{(j,i)\in\mathcal{E},\, y_j\neq\alpha} E_{ji}(y_j,\alpha)}_{C} - \underbrace{\sum_{(j,i)\in\mathcal{E},\, y_j\neq\alpha} E_{ji}(y_j,y_i)}_{D} .$$

Graph construction: n-links



n-links: for all $(i,j)\in\mathcal{E}$, where $i,j \in \mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}$

$$c(i,j) = E_{ij}(\alpha,y_j) + E_{ij}(y_i,\alpha) - E_{ij}(y_i,y_j) .$$

α-expansion algorithm *

Input: An *energy function* $E(\mathbf{y}) = \sum_{i\in\mathcal{V}} E_i(y_i) + \sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i,y_j)$ to be minimized, where E_{ij} is a **metric** for each $(i,j)\in\mathcal{E}$

Output: A local minimum $\mathbf{y}\in\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ of $E(\mathbf{y})$

1: Choose an arbitrary initial labeling $\mathbf{y}\in\mathcal{Y}$
2: $\hat{\mathbf{y}} \leftarrow \mathbf{y}$
3: **for all** $\alpha \in \mathcal{L}$ **do**
4: find $\hat{\mathbf{z}} \in \operatorname{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\hat{\mathbf{y}},\alpha)} E(\mathbf{z})$
5: $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{z}}$
6: **end for**
7: **if** $E(\hat{\mathbf{y}}) < E(\mathbf{y})$ **then**
8: $\mathbf{y} \leftarrow \hat{\mathbf{y}}$
9: Goto Step 2
10: **end if**

α-expansion is guaranteed to terminate in a finite number of cycles. This algorithm computes at least $|\mathcal{L}|$ graph cuts, which may take a lot of time, when the label space $\mathcal{L}$ is large.

Optimality *

The $\alpha-\beta$ swap does not guarantee any closeness to the global minimum. Nevertheless, the local minimum that the α-expansion algorithm will find is at most twice the global minimum $\mathbf{y}^*$.

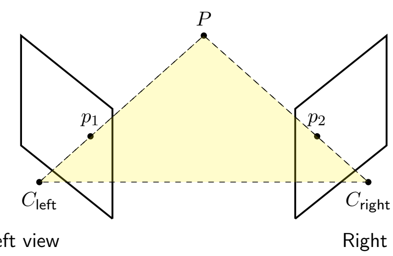
We have already assumed that E_{ij} is a metric for each $(i,j)\in\mathcal{E}$, hence $E_{ij}(\alpha,\beta)\neq 0$ for $\alpha\neq\beta\in\mathcal{L}$. Let us define

$$c = \max_{(i,j)\in\mathcal{E}} \left(\frac{\max_{\alpha\neq\beta\in\mathcal{L}} E_{ij}(\alpha,\beta)}{\min_{\alpha\neq\beta\in\mathcal{L}} E_{ij}(\alpha,\beta)} \right) .$$

Theorem 1. *Let $\hat{\mathbf{y}}$ be a local minimum when the expansion moves are allowed and $\mathbf{y}^*$ be the globally optimal solution. Then $E(\hat{\mathbf{y}}) \leqslant 2cE(\mathbf{y}^*)$.*

Stereo matching

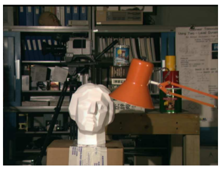
Stereo matching



Given two images (i.e. left and right), two observed 2D points p_1 and p_2 on the *left image* and *right image*, respectively, corresponding to a 3D point P in $\mathbb{R}^3$.
Note that P can be determined based on p_1 and p_2.
For more details you may refer to the course **Computer Vision II: Multiple View Geometry (IN2228)**.

Stereo matching

The goal is to reconstruct 3D points according to corresponding pixels.
We assume **rectified images** (i.e. the directions of the cameras are parallel), which means that the corresponding pixels are situated in **horizontal lines** according to some displacement.


Left view


Right view

Therefore, we need to search for corresponding points in the same row of both views. We also assume that the pixels p_1 and p_2 corresponding to P have similar intensities.

Energy function

We define $\mathcal{L} = \{1, 2, \ldots, D\}$ as the **label set**, i.e. set of possible *horizontal displacement* of pixels on the *right view*), where D is a constant.
Therefore the output domain $\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ and the *energy function* has the following form

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) \ ,$$

where $\mathbf{x}$ consists of the images (i.e. left and right view) denoted by $\mathbf{x}^{\text{left}}$ and $\mathbf{x}^{\text{right}}$, respectively.

Unary energies (a.k.a. **data terms**) E_i for all $i \in \mathcal{V}$ are defined as

$$E_i(y_i; \mathbf{x}) = \min(|x_i^{\text{left}} - x_{i+y_i}^{\text{right}}|^2, K)) \ ,$$

where K is a constant (e.g., $K = 20^2$).

Energy function

Pairwise energies (a.k.a. **smooth terms**) E_{ij} for all $(i, j) \in \mathcal{E}$ are defined as

$$E_{ij}(y_i, y_j; \mathbf{x}) = U(|x_i^{\text{left}} - x_j^{\text{left}}|) \cdot [\![y_i \neq y_j]\!] \ ,$$

where

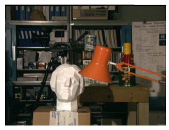
$$U(|x_i^{\text{left}} - x_j^{\text{left}}|) = \begin{cases} 2C, & \text{if } |x_i^{\text{left}} - x_j^{\text{left}}| \leqslant 5 \\ C, & \text{otherwise} \end{cases}$$

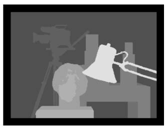
for some constant C.

Note the pairwise energies are defined by **weighted Potts-model**, which is a metric (see Exercise).

Results *


Left view


Right view


Ground truth


Result of $\alpha - \beta$ swap


Result of α-expansion

It is worth noting that α-expansion algorithm generally runs faster than $\alpha - \beta$ swap. There is optimality guarantee only for α-expansion algorithm, however, the two algorithms perform almost the same in many practical applications.

Summary *

- A binary energy function E consisting of up to pairwise functions is **regular**, if for each term E_{ij} for all $i < j$ satisfies

$$E_{ij}(0, 0) + E_{ij}(1, 1) \leqslant E_{ij}(0, 1) + E_{ij}(1, 0) \ .$$

- The *minimization of regular energy functions* can be achieved via *graph cut*.
- The *multi-label problem* for a finite label set $\mathcal{L}$

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) \ ,$$

can be approximately solved by applying
  - $\alpha - \beta$ `swap`, if E_{ij} is semi-metric;
  - α`-expansion`, if E_{ij} is metric.

In the **next lecture** we will learn about

- Linear programming relaxation for multi-label problem
- `Fast primal-dual algorithm`

Literature *

1. Vladimir Kolmogorov and Ramin Zabih. What energy functions can be minimized via graph cuts? *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(2):147–159, February 2004
2. Yuri Boykov, Olga Veksler, and Ramin Zabih. Fast approximate energy minimization via graph cuts. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 23(11):1222–1239, November 2001
3. Sebastian Nowozin and Christoph H. Lampert. Structured prediction and learning in computer vision. *Foundations and Trends in Computer Graphics and Vision*, 6(3–4), 2010