Probabilistic Graphical Models in Computer Vision (IN2329)

Csaba Domokos

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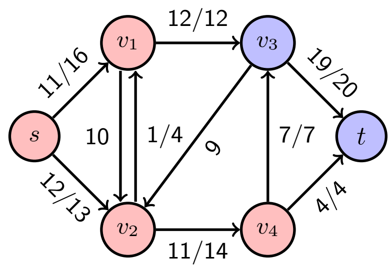
Agenda for today's lecture *

In the **previous lecture** we learnt about the minimum $s - t$ cut problem



Today we are going to learn about

- Exact solution for the binary image segmentation problem
- Approximate solutions for the multi-labeling problem:
  - α-expansion
  - $\alpha - \beta$ swap

Regular functions

Let us consider a function f of two binary variables, then f is called **regular**, if it satisfies the inequality

$$f(0,0) + f(1,1) \leqslant f(0,1) + f(1,0) \ .$$

Note that the **Potts-model** is *regular*, i.e.

$$[\![0 \neq 0]\!] + [\![1 \neq 1]\!] = 0 \leqslant 2 = [\![0 \neq 1]\!] + [\![1 \neq 0]\!] \ .$$

Regular energy functions

Let us consider an *energy function* E of n binary variables which can be written as the sum of functions of up to two variables, that is

$$E(y_1, \ldots, y_n) = \sum_i E_i(y_i) + \sum_{i<j} E_{ij}(y_i, y_j) .$$

E is *regular*, if each term E_{ij}

$$E_{ij}(0,0) + E_{ij}(1,1) \leqslant E_{ij}(0,1) + E_{ij}(1,0) .$$

If each term E_{ij} is *regular*, then it is possible to find the **global** minimum of E in *polynomial time* by solving a *minimum* $s-t$ *cut problem*.

Binary image segmentation

We have already seen that **binary image segmentation** can be reformulated as the minimization of an *energy function* $E : \{0,1\}^{\mathcal{V}} \times \mathcal{X} \to \mathbb{R}$:

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; x_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; x_i, x_j) .$$

where $\mathcal{V}$ corresponds to the output variables, i.e. the pixels, and $\mathcal{E}$ includes the pairs of neighboring pixels.

Assuming the probability densities p_b and p_f estimated for the background and the foreground, respectively, the **unary energies** E_i for all $i \in \mathcal{V}$ can be defined as

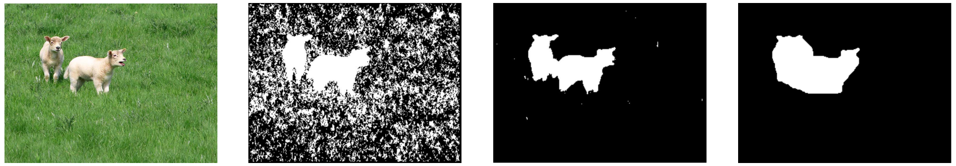
$$\begin{aligned} E_i(0, x_i) =& 0 \\ E_i(1, x_i) =& \frac{\log(p_b(x_i))}{\log(p_f(x_i))} . \end{aligned}$$

Contrast-sensitive Potts-model

The pairwise energy functions are defined as

$$E_{ij}(y_i, y_j; x_i, x_j) = w \exp(-\gamma \|x_i - x_j\|^2) [\![y_i \neq y_j]\!] \ ,$$

where $w \geqslant 0$ is a weighting factor. The parameter γ is the mean edge strength.



Original image | w is small | w is medium | w is high

Energy minimization

Remarks

Regularity is an extremely important property as is allows to minimize energy functions by making use of graph cut. Moreover, without the regularity constraint, the problem become intractable.

Let E_2 be a nonregular function of two binary variables. Then minimizing the energy function

$$E(y_1, \ldots, y_n) = \sum_i E_i(y_i) + \sum_{i<j} E_2(y_i, y_j) ,$$

where E_i are arbitrary functions of one binary variable, is NP-hard.

Multi-label problem

Multi-label problem

We define a label set $\mathcal{L} = \{1, 2, \ldots, L\}$, where L is a (finite) constant. Therefore the output domain is defined as $\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$. The *energy function* has the following form

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) ,$$

where $\mathbf{x}$ consists of an input image.

In order to ease to notation we will omit $\mathbf{x}$ and define the *energy function* simply as

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) .$$

Metric

A function $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}^+$ is called a **metric** if the following properties are satisfied:

1. **Identity of indiscernibles**: $d(\ell_1, \ell_2) = 0 \quad \Leftrightarrow \quad \ell_1 = \ell_2$ for all $\ell_1, \ell_2 \in \mathcal{L}$.
2. **Symmetry**: $d(\ell_1, \ell_2) = d(\ell_2, \ell_1)$ for all $\ell_1, \ell_2 \in \mathcal{L}$.
3. **Triangle inequality**: $d(\ell_1, \ell_3) \leqslant d(\ell_1, \ell_2) + d(\ell_2, \ell_3)$ for all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$.

Example: the **truncated absolute distance** $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x, y) = \min(K, |x - y|)$ is a *metric*, where K is some constant. (see Exercise)

If a function $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ satisfies the first two properties (i.e. identity of indiscernibles and symmetric), then it is called **semi-metric**.

Example: the **truncated quadratic function** $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x, y) = \min(K, |x - y|^2)$ is a *semi-metric*, where K is some constant. (see Exercise)

$\alpha - \beta$ swap

$\alpha - \beta$ **swap** changes the variables that are labeled as $\ell \in \{\alpha, \beta\}$. Each of these variables can choose either α or β. We introduce the following notation

$$\mathcal{Z}_{\alpha\beta}(\mathbf{y}, \alpha, \beta) = \{\mathbf{z} \in \mathcal{Y} : z_i = y_i, \ \text{if } y_i \notin \{\alpha, \beta\}, \ \text{otherwise } z_i \in \{\alpha, \beta\}\} \ .$$

The minimization of the *energy function* E can be reformulated as follows:

$$\begin{aligned}
\hat{\mathbf{z}} \in & \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} E(\mathbf{z}) = \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{i \in \mathcal{V}} E_i(z_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(z_i, z_j) \\
= & \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \Bigg[\underbrace{\sum_{i \in \mathcal{V},\, y_i \notin \{\alpha,\beta\}} E_i(y_i)}_{\text{constant}} + \underbrace{\sum_{i \in \mathcal{V},\, y_i \in \{\alpha,\beta\}} E_i(z_i)}_{\text{unary}} \\
& + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \notin \{\alpha,\beta\}}} E_{ij}(y_i, y_j)}_{\text{constant}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \in \{\alpha,\beta\},\, y_j \notin \{\alpha,\beta\}}} E_{ij}(z_i, y_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \notin \{\alpha,\beta\},\, y_j \in \{\alpha,\beta\}}} E_{ij}(y_i, z_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \in \{\alpha,\beta\}}} E_{ij}(z_i, z_j)}_{\text{pairwise}} \Bigg].
\end{aligned}$$

Local optimization

Let us consider $E_{ij}(z_i, z_j)$ for a given $(i, j) \in \mathcal{E}$:

	α	β
α	$E_{ij}(\alpha,\alpha)$	$E_{ij}(\alpha,\beta)$
β	$E_{ij}(\beta,\alpha)$	$E_{ij}(\beta,\beta)$

If we assume that $E_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is a **semi-metric** for each $(i, j) \in \mathcal{E}$, then

$$E_{ij}(\alpha,\alpha) + E_{ij}(\beta,\beta) = 0 \leqslant E_{ij}(\alpha,\beta) + E_{ij}(\beta,\alpha) = 2E_{ij}(\alpha,\beta) \ ,$$

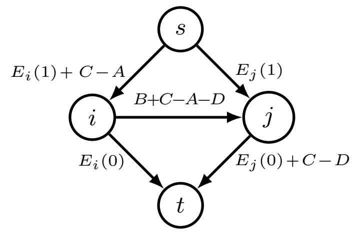
which means that E_{ij} is **regular** w.r.t. the labeling $\mathcal{Z}_{\alpha\beta}(\mathbf{y}, \alpha, \beta)$.

Graph construction for semi-metrics

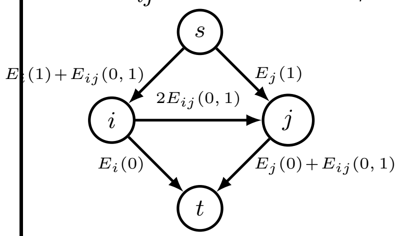
Let us consider the following *binary energy function*:

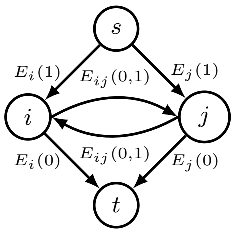
$$E(\mathbf{y}) = E_i(y_i) + E_j(y_j) + E_{ij}(y_i, y_j) \ ,$$

where E_{ij} is assumed to be a *semi-metric*.



Since E_{ij} is a *semi-metric*, we can construct a flow for $E(\mathbf{y})$ as follows:





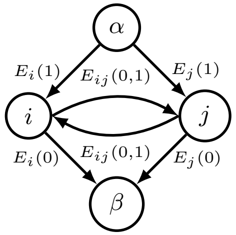
y_i	y_j	$E(\mathbf{y})$
0	0	$E_i(0)+E_j(0)$
0	1	$E_i(1)+E_j(0)+E_{ij}(1,0)$
1	0	$E_i(0)+E_j(1)+E_{ij}(1,0)$
1	1	$E_i(1)+E_j(1)$

Graph construction: t-links

We need to minimize the following **regular** *energy function*:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{\substack{i\in\mathcal{V}\\ y_i\in\{\alpha,\beta\}}} E_i(z_i) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\in\{\alpha,\beta\},\, y_j\notin\{\alpha,\beta\}}} E_{ij}(z_i,y_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\notin\{\alpha,\beta\},\, y_j\in\{\alpha,\beta\}}} E_{ij}(y_i,z_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i,y_j\in\{\alpha,\beta\}}} E_{ij}(z_i,z_j).$$

Based on construction applied for *binary image segmentation*, we can also define a *flow network* $(\mathcal{V}',\mathcal{E}',c,\alpha,\beta)$, where $\mathcal{V}'=\{\alpha,\beta\}\cup\{i\in\mathcal{V}: y_i\in\{\alpha,\beta\}\}$ and $\mathcal{E}'=\underbrace{\{(\alpha,i),(i,\beta): i\in\mathcal{V}'\backslash\{\alpha,\beta\}\}}_{\text{t-links}}\cup\underbrace{\{(i,j),(j,i)\mid i,j\in\mathcal{V}'\backslash\{\alpha,\beta\},(i,j)\in\mathcal{E}\}}_{\text{n-links}}$.

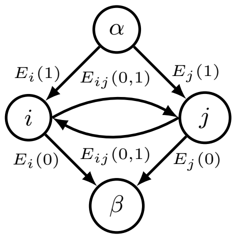


t-links: for all $i\in\mathcal{V}'\backslash\{\alpha,\beta\}$

$$c(\alpha,i) = E_i(\beta) + \sum_{(i,j)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ij}(\beta,y_j)\ .$$

$$c(i,\beta) = E_i(\alpha) + \sum_{(i,j)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ij}(\alpha,y_j)\ .$$

Graph construction: n-links



n-links: for all $(i,j)\in\mathcal{E}$, where $i,j\in\mathcal{V}'\backslash\{\alpha,\beta\}$

$$c(i,j) = c(j,i) = E_{ij}(\alpha,\beta)\ .$$

$\alpha - \beta$ swap algorithm

Input: An *energy function* $E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$ to be minimized, where E_{ij} is a **semi-metric** for each $(i, j) \in \mathcal{E}$
Output: A local minimum $\mathbf{y} \in \mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ of $E(\mathbf{y})$

1: Choose an arbitrary initial labeling $\mathbf{y} \in \mathcal{Y}$
2: $\hat{\mathbf{y}} \leftarrow \mathbf{y}$
3: **for all** $(\alpha, \beta) \in \mathcal{L} \times \mathcal{L}$ **do**
4:     find $\hat{\mathbf{z}} \in \operatorname{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\hat{\mathbf{y}}, \alpha, \beta)} E(\mathbf{z})$
5:     $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{z}}$
6: **end for**
7: **if** $E(\hat{\mathbf{y}}) < E(\mathbf{y})$ **then**
8:     $\mathbf{y} \leftarrow \hat{\mathbf{y}}$
9:     Goto Step 2
10: **end if**

$\alpha - \beta$ swap algorithm is guaranteed to terminate in a finite number of cycles. This algorithm computes at least $|\mathcal{L}|^2$ graph cuts, which may take a lot of time, even for moderately large label spaces.

α-expansion

α-expansion allows each variable either to keep its current label or to change it to the label $\alpha \in \mathcal{L}$. We introduce the following notation

$$\mathcal{Z}_\alpha(\mathbf{y}, \alpha) = \{\mathbf{z} \in \mathcal{Y} : z_i \in \{y_i, \alpha\} \text{ for all } i \in \mathcal{V}\} .$$

The minimization of the *energy function* E can be reformulated as follows:

$$\begin{aligned}
\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} E(\mathbf{z}) &= \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} \sum_{i\in\mathcal{V}} E_i(z_i) + \sum_{(i,j)\in\mathcal{E}} E_{ij}(z_i, z_j) \\
&= \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} \Bigg[\underbrace{\sum_{i\in\mathcal{V},\, y_i=\alpha} E_i(\alpha)}_{\text{constant}} + \underbrace{\sum_{i\in\mathcal{V},\, y_i\neq\alpha} E_i(z_i)}_{\text{unary}} \\
&\quad + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i=\alpha,\, y_j=\alpha}} E_{ij}(\alpha, \alpha)}_{\text{constant}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i=\alpha,\, y_j\neq\alpha}} E_{ij}(\alpha, z_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j=\alpha}} E_{ij}(z_i, \alpha)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j\neq\alpha}} E_{ij}(z_i, z_j)}_{\text{pairwise}} \Bigg] .
\end{aligned}$$

Local optimization

Let us consider $E_{ij}(z_i, z_j)$ for a given $(i,j) \in \mathcal{E}$:

	α	y_j
α	$E_{ij}(\alpha,\alpha)$	$E_{ij}(\alpha,y_j)$
y_i	$E_{ij}(y_i,\alpha)$	$E_{ij}(y_i,y_j)$

If we assume that $E_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is a **metric** for each $(i,j) \in \mathcal{E}$, then

$$E_{ij}(\alpha,\alpha) + E_{ij}(y_i,y_j) = E_{ij}(y_i,y_j) \leqslant E_{ij}(y_i,\alpha) + E_{ij}(\alpha,y_j) \; ,$$

which means that E_{ij} is **regular** w.r.t. the labeling $\mathcal{Z}_\alpha(\mathbf{y},\alpha)$.

Energy reformulation

We have the following energy minimization problem:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} \underbrace{\sum_{\substack{i\in\mathcal{V}\\ y_i\neq\alpha}} E_i(z_i) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i=\alpha,\, y_j\neq\alpha}} E_{ij}(\alpha,z_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j=\alpha}} E_{ij}(z_i,\alpha)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j\neq\alpha}} E_{ij}(z_i,z_j)}_{\text{pairwise}} \; .$$

We introduce the following notation

$$\mathcal{N}_\alpha(i) := \{j \in \mathcal{V} : y_j = \alpha\} \cap \{j \in \mathcal{V} : (i,j) \in \mathcal{E} \text{ or } (j,i) \in \mathcal{E})\} \; .$$

We can rewrite the *energy function* as:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} \underbrace{\sum_{\substack{i\in\mathcal{V}\\ y_i\neq\alpha}} E_i(z_i) + \sum_{\substack{i\in\mathcal{V},\, j\in\mathcal{N}_\alpha(i)\\ y_i\neq\alpha}} E_{ij}(z_i,\alpha)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j\neq\alpha}} E_{ij}(z_i,z_j)}_{\text{pairwise}} \; ,$$

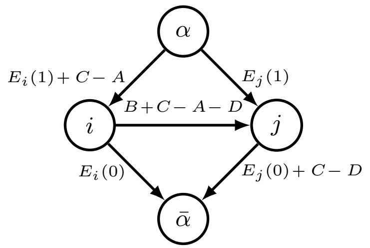
where $E_{ji}(z_j,\alpha) := E_{ij}(\alpha,z_j) = E_{ij}(z_j,\alpha)$ is assumed for each $(i,j) \in \mathcal{E}$.

Graph construction: t-links

We need to minimize the following **regular** *energy function*:

$$\hat{\mathbf{z}} \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_\alpha(\mathbf{y},\alpha)} \sum_{\substack{i\in\mathcal{V}\\ y_i\neq\alpha}} E_i(z_i) + \sum_{\substack{i\in\mathcal{V},\, j\in\mathcal{N}_\alpha(i)\\ y_i\neq\alpha}} E_{ij}(z_i,\alpha) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\neq\alpha,\, y_j\neq\alpha}} E_{ij}(z_i,z_j)\ .$$

Based on construction applied for *binary image segmentation*, we can also define a *flow network* $(\mathcal{V}',\mathcal{E}',c,\alpha,\bar{\alpha})$, where $\mathcal{V}'=\{\alpha,\bar{\alpha}\}\cup\{i\in\mathcal{V}: y_i\neq\alpha\}$ and $\mathcal{E}'=\underbrace{\{(\alpha,i),(i,\bar{\alpha}): i\in\mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}\}}_{\text{t-links}}\cup\underbrace{\{(i,j)\in\mathcal{E}: i,j\in\mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}\}}_{\text{n-links}}$.

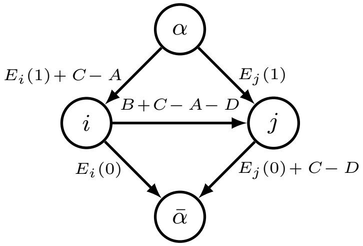


t-links: for all $i\in\mathcal{V}'\backslash\{\alpha,\bar{\alpha}\}$

$$c(\alpha,i) = E_i(y_i) + \sum_{j\in\mathcal{N}_\alpha(i)} E_{ij}(y_i,\alpha) + \sum_{(i,j)\in\mathcal{E},\, y_j\neq\alpha} E_{ij}(y_i,\alpha)\ .$$

$$c(i,\bar{\alpha}) = E_i(\alpha) + \sum_{(i,j)\in\mathcal{E},\, y_j\neq\alpha} E_{ij}(y_i,\alpha) - E_{ij}(y_i,y_j)\ .$$

Graph construction: t-links



t-links: for all $i \in \mathcal{V}' \backslash \{\alpha, \bar{\alpha}\}$

$$c(\alpha, i) = E_i(y_i) + \sum_{j \in \mathcal{N}_\alpha(i)} E_{ij}(y_i, \alpha) + \sum_{(i,j) \in \mathcal{E},\, y_j \neq \alpha} E_{ij}(y_i, \alpha) \; .$$

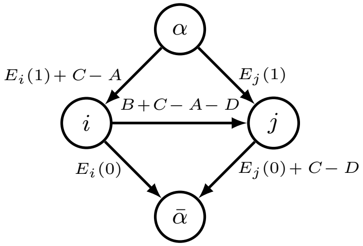
$$c(i, \bar{\alpha}) = E_i(\alpha) + \sum_{(i,j) \in \mathcal{E},\, y_j \neq \alpha} E_{ij}(y_i, \alpha) - E_{ij}(y_i, y_j) \; .$$

Or equivalently,

$$c(\alpha, i) = E_i(y_i) + \sum_{j \in \mathcal{N}_\alpha(i)} E_{ij}(y_i, \alpha) + \sum_{(i,j) \in \mathcal{E},\, y_j \neq \alpha} E_{ij}(y_i, y_j) \; .$$

$$c(i, \bar{\alpha}) = E_i(\alpha) \; .$$

Graph construction: n-links



n-links: for all $(i, j) \in \mathcal{E}$, where $i, j \in \mathcal{V}' \backslash \{\alpha, \bar{\alpha}\}$

$$c(i, j) = E_{ij}(\alpha, y_j) + E_{ij}(y_i, \alpha) - E_{ij}(y_i, y_j) \ .$$

α-expansion algorithm

Input: An *energy function* $E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$ to be minimized, where E_{ij} is a **metric** for each $(i, j) \in \mathcal{E}$
Output: A local minimum $\mathbf{y} \in \mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ of $E(\mathbf{y})$

1: Choose an arbitrary initial labeling $\mathbf{y} \in \mathcal{Y}$
2: $\hat{\mathbf{y}} \leftarrow \mathbf{y}$
3: **for all** $\alpha \in \mathcal{L}$ **do**
4:     find $\hat{\mathbf{z}} \in \operatorname{argmin}_{\mathbf{z} \in \mathcal{Z}_\alpha(\hat{\mathbf{y}}, \alpha)} E(\mathbf{z})$
5:     $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{z}}$
6: **end for**
7: **if** $E(\hat{\mathbf{y}}) < E(\mathbf{y})$ **then**
8:     $\mathbf{y} \leftarrow \hat{\mathbf{y}}$
9:     Goto Step 2
10: **end if**

α-expansion is guaranteed to terminate in a finite number of cycles. This algorithm computes at least $|\mathcal{L}|$ graph cuts, which may take a lot of time, when the label space $\mathcal{L}$ is large.

Optimality

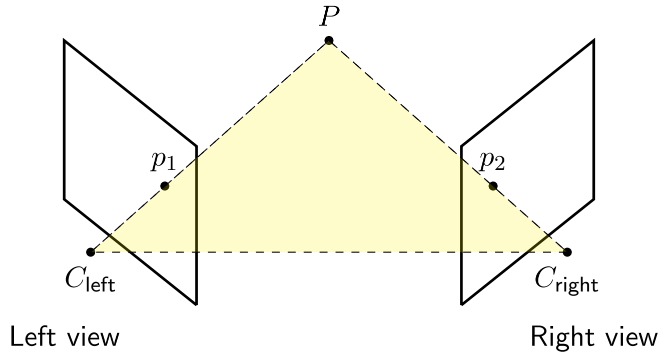
The $\alpha - \beta$ swap does not guarantee any closeness to the global minimum. Nevertheless, the local minimum that the α-expansion algorithm will find is at most twice the global minimum $\mathbf{y}^*$.

We have already assumed that E_{ij} is a metric for each $(i,j) \in \mathcal{E}$, hence $E_{ij}(\alpha,\beta) \neq 0$ for $\alpha \neq \beta \in \mathcal{L}$. Let us define

$$c = \max_{(i,j)\in\mathcal{E}} \left(\frac{\max_{\alpha\neq\beta\in\mathcal{L}} E_{ij}(\alpha,\beta)}{\min_{\alpha\neq\beta\in\mathcal{L}} E_{ij}(\alpha,\beta)} \right) .$$

Theorem 1. *Let $\hat{\mathbf{y}}$ be a local minimum when the expansion moves are allowed and $\mathbf{y}^*$ be the globally optimal solution. Then $E(\hat{\mathbf{y}}) \leqslant 2cE(\mathbf{y}^*)$.*

Stereo matching

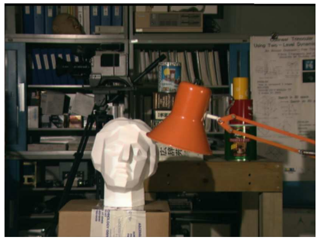


Given two images (i.e. left and right), two observed 2D points p_1 and p_2 on the *left image* and *right image*, respectively, corresponding to a 3D point P in $\mathbb{R}^3$. Note that P can be determined based on p_1 and p_2.
For more details you may refer to the course **Computer Vision II: Multiple View Geometry (IN2228)**.

Stereo matching

The goal is to reconstruct 3D points according to corresponding pixels.
We assume **rectified images** (i.e. the directions of the cameras are parallel), which means that the corresponding points situated in **horizontal lines** according to some displacement.


Left view


Right view

Therefore, we need to search for corresponding points in the same row of both views. We also assume that the pixels p_1 and p_2 corresponding to P have similar intensities.

Energy function

We define $\mathcal{L} = \{1, 2, \ldots, D\}$ as the **label set**, i.e. set of possible *horizontal displacement* of pixels on the *right view*), where D is a constant.
Therefore the output domain $\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ and the *energy function* has the following form

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) ,$$

where $\mathbf{x}$ consists of the images (i.e. left and right view) denoted by $\mathbf{x}^{\text{left}}$ and $\mathbf{x}^{\text{left}}$, respectively.

Unary energies (a.k.a. **data terms**) E_i for all $i \in \mathcal{V}$ are defined as

$$E_i(y_i; \mathbf{x}) = \min(|x_i^{\text{left}} - x_{i+y_i}^{\text{right}}|^2, K)) ,$$

where K is a constant (e.g., $K = 20^2$).

Energy function

Pairwise energies (a.k.a. **smooth terms**) E_{ij} for all $(i,j) \in \mathcal{E}$ are defined as

$$E_{ij}(y_i, y_j; \mathbf{x}) = U(|x_i^{\text{left}} - x_j^{\text{left}}|) \cdot [\![y_i \neq y_j]\!] \ ,$$

where

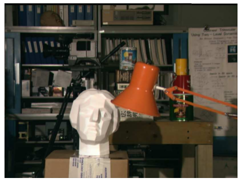
$$U(|x_i^{\text{left}} - x_j^{\text{left}}|) = \begin{cases} 2C, & \text{if } |x_i^{\text{left}} - x_j^{\text{left}}| \leqslant 5 \\ C, & \text{otherwise} \end{cases}$$

for some constant C.

Note the pairwise energies are defined by **weighted Potts-model**, which is a metric (see Exercise).

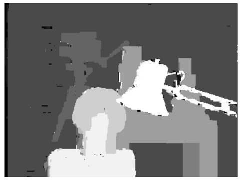
Results

Left view

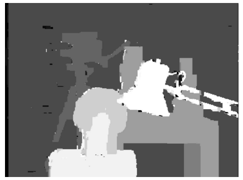


Right view

Ground truth



Result of $\alpha - \beta$ swap



Result of α-expansion

It is worth noting that α-expansion algorithm generally runs faster than $\alpha - \beta$ swap. There is optimality guarantee only for α-expansion algorithm, however, the two algorithms perform almost the same in many practical applications.

Literature *

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