Probabilistic Graphical Models in Computer Vision (IN2329)

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Agenda for today's lecture *

Let us consider an *undirected graphical model* given by $G = (\mathcal{V}, \mathcal{E})$, which takes values from an **arbitrary** (finite) label set $\mathcal{L}$. More specially, assume that the corresponding *energy function* $E : \mathcal{L}^{\mathcal{V}} \to \mathbb{R}$ is given by

$$E(\mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(\mathbf{x}_i) + \sum_{(i,j) \in \mathcal{E}} w_{ij} \cdot d(\mathbf{x}_i, \mathbf{x}_j) \ ,$$

where E_i stands for a *unary energy function*, $w_{ij} \in \mathbb{R}$ are *weighting factors*, and d is a *metric* or a *semi-metric* (i.e. the triangle inequality is not necessary satisfied).

In the **previous lecture** we learnt about the move making algorithms (i.e. $\alpha - \beta$ swap, α-expansion) as a possible way to *approximately* solve this problem.

Today we are going to learn about the `FastPD algorithm`, which provides an approximate solution via *linear programming*.

Equivalent integer linear program

We are generally interested to find a *MAP labelling* $\mathbf{x}^*$:

$$\mathbf{x}^* \in \operatorname*{argmin}_{\mathbf{x}\in\mathcal{L}^{|\mathcal{V}|}} E(\mathbf{x}) = \operatorname*{argmin}_{\mathbf{x}\in\mathcal{L}^{|\mathcal{V}|}} \left\{ \sum_{i\in\mathcal{V}} E_i(x_i) + \sum_{(i,j)\in\mathcal{E}} w_{ij} \cdot d(x_i, x_j) \right\} .$$

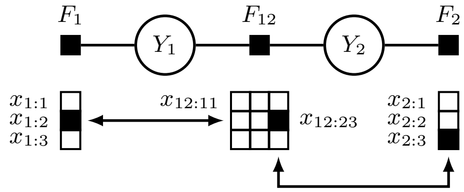
This can be equivalently written as an **integer linear program** (ILP):

$$\begin{aligned} \min_{x_{i:\alpha}, x_{ij:\alpha\beta}} & \sum_{i\in\mathcal{V}} \sum_{\alpha\in\mathcal{L}} E_i(\alpha) x_{i:\alpha} + \sum_{(i,j)\in\mathcal{E}} w_{ij} \sum_{\alpha,\beta\in\mathcal{L}} d(\alpha,\beta) x_{ij:\alpha\beta} \\ \text{subject to} \quad & \textstyle\sum_{\alpha\in\mathcal{L}} x_{i:\alpha} = 1 \quad \forall i \in \mathcal{V} \\ & \textstyle\sum_{\alpha\in\mathcal{L}} x_{ij:\alpha\beta} = x_{j:\beta} \quad \forall \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & \textstyle\sum_{\beta\in\mathcal{L}} x_{ij:\alpha\beta} = x_{i:\alpha} \quad \forall \alpha \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & x_{i:\alpha}, x_{ij:\alpha\beta} \in \mathbb{B} \quad \forall \alpha, \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \end{aligned}$$

$x_{i:\alpha}$ indicates whether vertex i is assigned label α, while $x_{ij:\alpha\beta}$ indicates whether (neighboring) vertices i, j are assigned labels α, β, respectively.

Interpretation of the constraints

Let us assume that $\mathcal{L} = \{1, 2, 3\}$ and consider the following factor graph example:



Uniqueness: The constraints $\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} = 1$ for all $i \in \mathcal{V}$ simply express the fact that each vertex must receive exactly one label.

Consistency: The constraints

$$\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} = x_{j:\beta} \quad \text{and} \quad \sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} = x_{i:\alpha} \quad \forall \alpha, \beta \in \mathcal{L}\ , (i,j) \in \mathcal{E}$$

maintain consistency between variables, i.e. if $x_{i:\alpha} = 1$ and $x_{j:\beta} = 1$ holds true, then these constraints force $x_{ij:\alpha\beta} = 1$ to hold true as well.

Primal-dual LP

LP relaxation *

The ILP defined before is in general NP-hard. Therefore we deal with the **LP relaxation** of our ILP. The relaxed LP can be written in *standard form* as follows:

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \langle \mathbf{c}, \mathbf{x} \rangle$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\ .$$

LP relaxation: cost function *

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \langle \mathbf{c}, \mathbf{x} \rangle \qquad \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0} .$$

We may write $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}^T$, where

$$\mathbf{x}_1 = \begin{bmatrix} x_{1:1} & \cdots & x_{1:3} & x_{2:1} & \cdots & x_{2:3} \end{bmatrix}^T \in \mathbb{R}^{mn} ,$$

where $n = |\mathcal{V}|$ and $m = |\mathcal{L}|$, and

$$\mathbf{x}_2 = \begin{bmatrix} x_{12:11} & \cdots & x_{12:13} & \cdots & x_{12:31} & \cdots & x_{12:33} \end{bmatrix}^T \in \mathbb{R}^{|\mathcal{E}|m^2} .$$

Similarly, we can write $\mathbf{c} = \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T \end{bmatrix}^T$, where

$$\mathbf{c}_1 = \begin{bmatrix} E_1(1) & \cdots & E_1(3) & E_2(1) & \cdots & E_2(3) \end{bmatrix}^T \in \mathbb{R}^{mn}$$
$$\mathbf{c}_2 = \begin{bmatrix} w_{12}d(1,1) & \cdots & w_{12}d(1,3) & \cdots & w_{12}d(3,1) & \cdots & w_{12}d(3,3) \end{bmatrix}^T \in \mathbb{R}^{|\mathcal{E}|m^2}.$$

Therefore, $\langle \mathbf{c}, \mathbf{x} \rangle = \langle \mathbf{c}_1, \mathbf{x}_1 \rangle + \langle \mathbf{c}_2, \mathbf{x}_2 \rangle$.

LP relaxation: uniqueness constraints *

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \langle \mathbf{c}, \mathbf{x} \rangle \qquad \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0} .$$

We can write the (uniqueness) constraints $\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} = 1$ for all $i \in \mathcal{V}$ as

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}_{11}} \begin{bmatrix} x_{1:1} \\ \vdots \\ x_{2:3} \end{bmatrix} = \mathbf{A}_{11}\mathbf{x}_1 = \mathbf{1}_n =: \mathbf{b}_1 ,$$

where $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all-ones.

LP relaxation: consistency constraints *

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \langle \mathbf{c}, \mathbf{x} \rangle \qquad \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0} \;.$$

The (consistency) constraints $\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} = x_{j:\beta} \;\Leftrightarrow\; -x_{j:\beta} + \sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} = 0$ and $\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} = x_{i:\alpha} \;\Leftrightarrow\; -x_{i:\alpha} + \sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} = 0$ can be expressed as

$$\left[\begin{array}{cccccc|ccccccccc} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{bmatrix} x_{1:1} \\ \vdots \\ x_{2:3} \\ x_{12:11} \\ \vdots \\ x_{12:33} \end{bmatrix} = \mathbf{0},$$

$$\left[\begin{array}{c|c} \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{0}_{2|\mathcal{E}|m} =: \mathbf{b}_2 \;.$$

LP relaxation: constraints *

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta}} \langle \mathbf{c}, \mathbf{x} \rangle \qquad \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0} \;.$$

We can write all the constraints in a matrix-vector notation as follows.

$$\mathbf{A}\mathbf{x} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{0}_{n \times |\mathcal{E}|m^2} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_{2|\mathcal{E}|m} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \mathbf{b} \;.$$

Hence, $\mathbf{A} \in \mathbb{R}^{n+2|\mathcal{E}|m \times mn+|\mathcal{E}|m^2}$ is a **sparse matrix** with elements -1,0 and 1, furthermore $\mathbf{b} \in \mathbb{R}^{n+2|\mathcal{E}|m}$, where the first mn elements are equal to one and the others are equal to zero.

Primal-dual LP

Consider a linear program (given in **standard form**):

$$\min_{\mathbf{x}\in\mathbb{R}^n} \langle \mathbf{c}, \mathbf{x} \rangle$$
$$\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\,,$$

for a *constraint matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$, a *constraint vector* $\mathbf{b} \in \mathbb{R}^m$ and a *cost vector* $\mathbf{c} \in \mathbb{R}^n$.

The *dual LP* is defined as

$$\max_{\mathbf{y}\in\mathbb{R}^m} \langle \mathbf{b}, \mathbf{y} \rangle$$
$$\text{subject to } \mathbf{A}^T\mathbf{y} \leqslant \mathbf{c}\,.$$

For feasible solutions $\mathbf{x}$ and $\mathbf{y}$ **weak duality** holds:

$$\langle \mathbf{b}, \mathbf{y} \rangle = \mathbf{b}^T\mathbf{y} = \mathbf{x}^T(\mathbf{A}^T\mathbf{y}) = (\mathbf{y}^T\mathbf{A})\mathbf{x} \leqslant \mathbf{c}^T\mathbf{x} = \langle \mathbf{c}, \mathbf{x} \rangle\,.$$

Dual LP

$$\max_{y_i, y_{ij:\alpha}, y_{ji:\beta}} \langle \mathbf{b}, \mathbf{y} \rangle \qquad \text{subject to } \mathbf{A}^T \mathbf{y} \leqslant \mathbf{c} \,.$$

Note that the dual variables y_i for all $i \in \mathcal{V}$ and $y_{ij:\alpha}$, $y_{ji:\beta}$ for all $(i,j) \in \mathcal{E}$, $\alpha, \beta \in \mathcal{L}$ correspond to the constraints of the primal LP.

We can write $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \mathbf{y}_3^T \end{bmatrix}^T$, where $\mathbf{y}_1 = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T \in \mathbb{R}^n$, and $\mathbf{y}_2 \in \mathbb{R}^{|\mathcal{E}|m}$ and $\mathbf{y}_3 \in \mathbb{R}^{|\mathcal{E}|m}$ are the vectors consisting of the variables $y_{ji:\beta}$ and $y_{ij:\alpha}$ in the same order as it is defined in the case of the primal LP.

The cost function results in

$$\langle \mathbf{b}, \mathbf{y} \rangle = \langle \mathbf{b}_1, \mathbf{y}_1 \rangle + \langle \mathbf{b}_2, \begin{bmatrix} \mathbf{y}_2^T & \mathbf{y}_3^T \end{bmatrix}^T \rangle = \langle \mathbf{1}_n, \mathbf{y}_1 \rangle = \sum_{i=1}^n y_i \,.$$

The constraints $\mathbf{A}^T \mathbf{y} \leqslant \mathbf{c}$ are given by

$$\mathbf{A}^T \mathbf{y} = \left[\begin{array}{c|c} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \hline \mathbf{0}_{|\mathcal{E}|m^2 \times n} & \mathbf{A}_{22}^T \end{array} \right] \mathbf{y} \leqslant \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} = \mathbf{c} \,.$$

Dual LP *

$$\max_{y_i; y_{ij:\alpha}; y_{ji:\beta}} \langle \mathbf{1}_n, \mathbf{y}_1 \rangle$$

$$\text{subject to} \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \hline \mathbf{0}_{|\mathcal{E}|m^2 \times n} & \mathbf{A}_{22}^T \end{bmatrix} \mathbf{y} \leqslant \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} .$$

Or equivalently, we can formulate the dual LP as

$$\begin{aligned} \max_{y_i; y_{ij:\alpha}; y_{ji:\beta}} & \sum_{i \in \mathcal{V}} y_i \\ \text{subject to} \quad & y_i - \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:\alpha} \leqslant E_i(\alpha) && \forall i \in \mathcal{V}, \alpha \in \mathcal{L} \\ & y_{ij:\alpha} + y_{ji:\beta} \leqslant w_{ij} d(\alpha, \beta) && \forall (i,j) \in \mathcal{E}, \alpha, \beta \in \mathcal{L} \end{aligned}$$

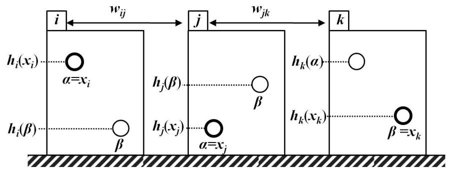
An intuitive view of the dual variables

We will refer to $x_i \in \mathcal{L}$ as the **active label** for a given the vertex $i \in \mathcal{V}$.

For each vertex we have a different copy of all labels in $\mathcal{L}$. It is assumed that all these labels represent **balls** floating at certain heights relative to a *reference plane*.

For this sake we introduce **height variables** defined as

$$h_i(\alpha) \triangleq E_i(\alpha) + \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:\alpha} .$$



The constraints $y_i - \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} \leqslant E_i(\alpha)$ can be equivalently written as

$$y_i \leqslant E_i(\alpha) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} = h_i(\alpha) \qquad \forall i \in \mathcal{V}, \alpha \in \mathcal{L} .$$

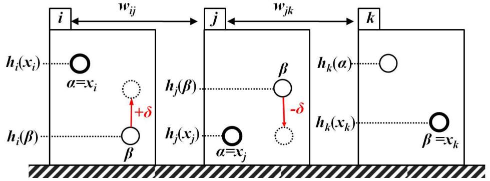
Since our objective is to maximize $\sum_{i \in \mathcal{V}} y_i$, the following relation holds

$$y_i = \min_{\alpha \in \mathcal{L}} h_i(\alpha) \qquad \forall i \in \mathcal{V} .$$

Balance variables and load

We will refer to the variables $y_{ij:\alpha}$, $y_{ji:\beta}$ as **balance variables**. Specially, the pair of $y_{ij:\alpha}$, $y_{ji:\alpha}$ is called **conjugate balance variables**.

The *balls* are not static, but may move in pairs through updating pairs of *conjugate balance variables* as $h_i(\alpha) = \varphi_i(\alpha) + \sum_{j \in \mathcal{V}, (i,j) \in \mathcal{E}} y_{ij:\alpha}$. Therefore, the role of *balance variables* is to raise or lower labels.



It is due to $y_{ij:\alpha} + y_{ji:\alpha} \leqslant w_{ij} d(\alpha, \alpha) = 0 \quad \Rightarrow \quad y_{ij:\alpha} \leqslant -y_{ji:\alpha}$.

We will call the variables $y_{ij:x_i}$ as **active balance variable** and use the following notation for the **"load"** between neighbors i, j, defined as

$$\text{load}_{ij} = y_{ij:x_i} + y_{ji:x_j} \ .$$

Primal-dual LP for multi-label problem

The (relaxed) primal LP:

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta} \geqslant 0} \sum_{i \in \mathcal{V}} \sum_{\alpha \in \mathcal{L}} E_i(\alpha) x_{i:\alpha} + \sum_{(i,j) \in \mathcal{E}} w_{ij} \sum_{\alpha, \beta \in \mathcal{L}} d(\alpha, \beta) x_{ij:\alpha\beta}$$

subject to

$$\begin{aligned} \textstyle\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} &= 1 && \forall i \in \mathcal{V} \\ \textstyle\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{j:\beta} && \forall \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \\ \textstyle\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{i:\alpha} && \forall \alpha \in \mathcal{L}, (i,j) \in \mathcal{E} \end{aligned}$$

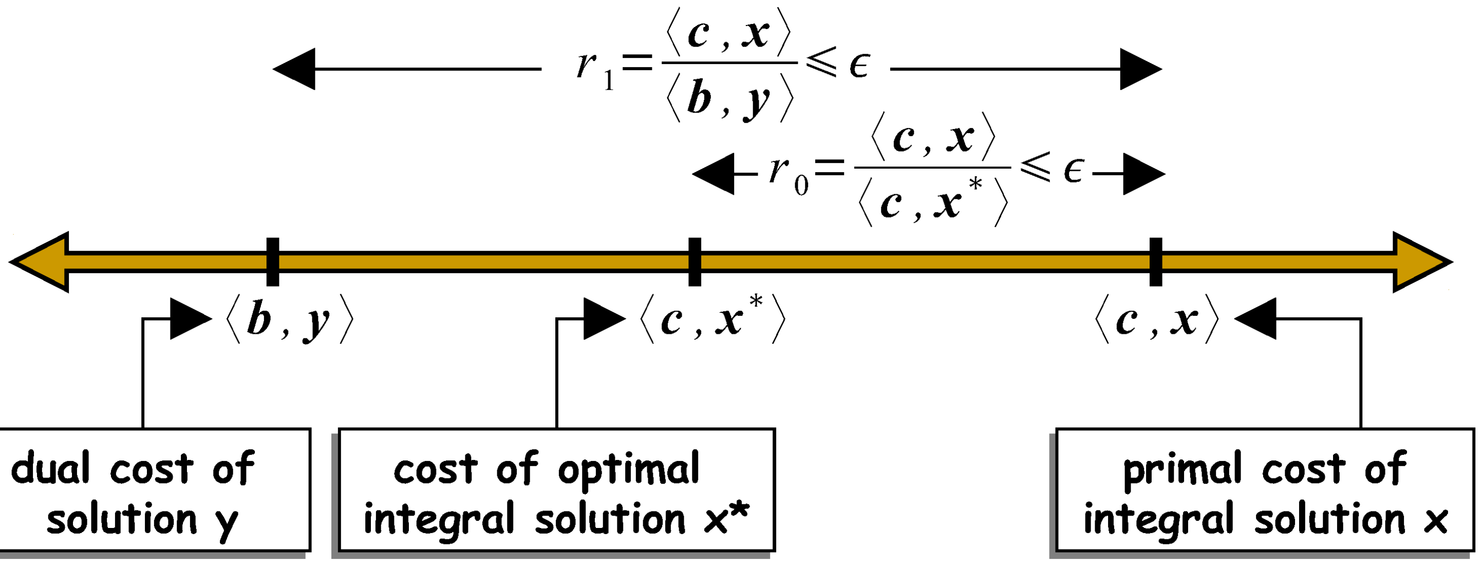
The dual LP:

$$\max_{y_i, y_{ij:\alpha}, y_{ji:\beta}} \sum_{i \in \mathcal{V}} y_i$$

subject to

$$\begin{aligned} y_i - \sum_{j \in \mathcal{V} : (i,j) \in \mathcal{E}} y_{ij:\alpha} &\leqslant E_i(\alpha) && \forall i \in \mathcal{V}, \alpha \in \mathcal{L} \\ y_{ij:\alpha} + y_{ji:\beta} &\leqslant w_{ij} d(\alpha, \beta) && \forall (i,j) \in \mathcal{E}, \alpha, \beta \in \mathcal{L} \end{aligned}$$

Primal-dual principle



Theorem 1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are integral-primal and dual feasible solutions satisfying:*

$$\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$$

for $\epsilon \geqslant 1$, *then* $\mathbf{x}$ *is an* **ϵ-approximation** *to the optimal integral solution* $\mathbf{x}^*$, *that is*

$$\langle \mathbf{c}, \mathbf{x}^* \rangle \leqslant \langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle \leqslant \epsilon \langle \mathbf{c}, \mathbf{x}^* \rangle .$$

The relaxed complementary slackness

One way to estimate a pair $(\mathbf{x}, \mathbf{y})$ satisfying the fundamental inequality

$$\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$$

relies the **complementary slackness principle**.

Theorem 2. *If the pair $(\mathbf{x}, \mathbf{y})$ of integral-primal and dual feasible solutions satisfies the so-called* **relaxed primal complementary slackness conditions***:*

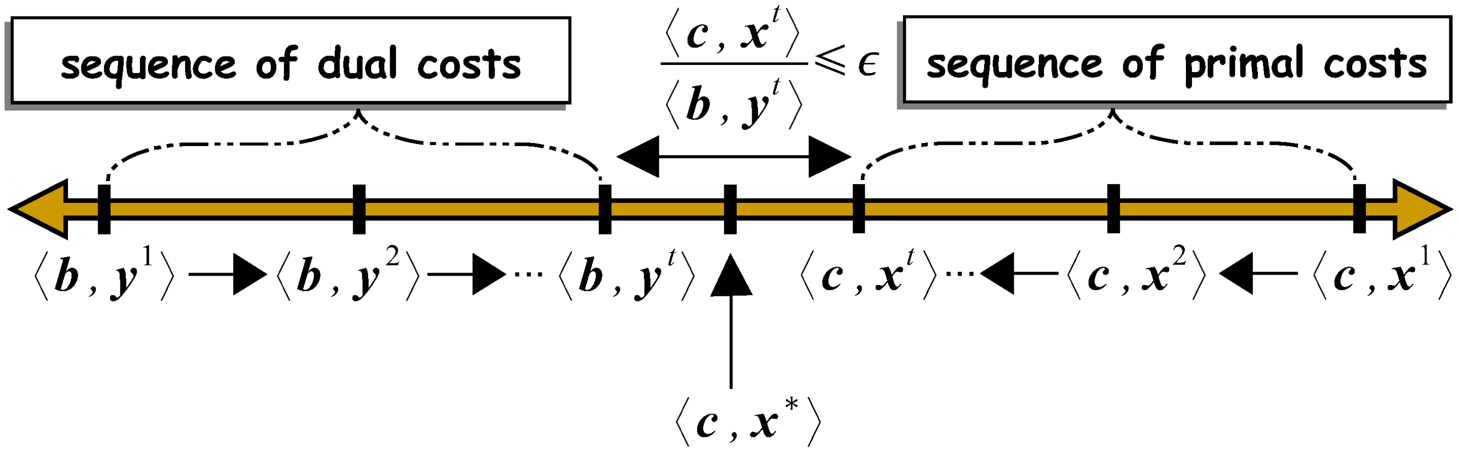
$$\forall j : \ (x_j > 0) \quad \Rightarrow \quad \sum_i a_{ij} y_i \geqslant \frac{c_j}{\epsilon_j} \ ,$$

then $(\mathbf{x}, \mathbf{y})$ also satisfies $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ with $\epsilon = \max_j \epsilon_j$ and therefore $\mathbf{x}$ is an ϵ-approximation to the optimal integral solution $\mathbf{x}^*$.

Proof. Exercise. □

We aim to satisfy relaxed complementary slackness conditions in order to achieve an ϵ-approximation solution.

Primal-dual schema



Typically, primal-dual ϵ-approximation algorithms construct a sequence $(\mathbf{x}^k, \mathbf{y}^k)_{k=1,\ldots,t}$ of primal and dual solutions until the elements $\mathbf{x}^t$, $\mathbf{y}^t$ of the last pair are both **feasible** and **satisfy the relaxed primal complementary slackness conditions**, hence the condition $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ will be also fulfilled.

Pseudo-code of the FastPD algorithm *

```
1: [x y] ←Init_Primals_Duals()
2: labelChange ←false
3: for all α ∈ ℒ do                                          ▷ α-iteration
4:     y ←PreEdit_Duals(α, x, y)
5:     [x' y'] ←Update_Duals_Primals(α, x, y)
6:     y' ←PostEdit_Duals(α, x', y')
7:     if x' ≠ x then
8:         labelChange ←true
9:     end if
10:    x ← x' and y ← y'
11: end for
12: if labelChange then
13:     goto 2
14: end if
15: y^fit ←Dual_Fit(y)
```

Complementary slackness conditions *

From now on, in case of Algorithm `PD1`, we only assume that $d(\alpha,\beta) = 0 \Leftrightarrow \alpha = \beta$, and $d(\alpha,\beta) \geqslant 0$ (i.e. d is a semi-metric).

The *complementary slackness conditions* reduces to

$$y_i - \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i} \geqslant \frac{E_i(x_i)}{\epsilon_1} \quad \Rightarrow \quad y_i \geqslant \frac{E_i(x_i)}{\epsilon_1} + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i}$$

$$y_{ij:x_i} + y_{ji:x_j} \geqslant \frac{w_{ij} d(x_i, x_j)}{\epsilon_2}$$

for specific values of $\epsilon_1, \epsilon_2 \geqslant 1$.

If $x_i = x_j = \alpha$ for neighboring pairs $(i,j) \in \mathcal{E}$, then

$$0 = w_{ij} d(\alpha,\alpha) \geqslant y_{ij:\alpha} + y_{ji:\alpha} \geqslant \frac{w_{ij} d(\alpha,\alpha)}{\epsilon_2} = 0\ ,$$

therefore we get that $y_{ij:\alpha} = -y_{ji:\alpha}$.

Complementary slackness conditions *

We have already known that $y_i = \min_{\alpha \in \mathcal{L}} h_i(\alpha)$. If $\epsilon_1 = 1$, then we get

$$y_i \geqslant \frac{E_i(x_i)}{\epsilon_1} + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:x_i} = h_i(x_i) \ .$$

Therefore

$$h_i(x_i) = \min_{\alpha \in \mathcal{L}} h_i(\alpha) \ , \tag{1}$$

which means that, at each vertex, **the active label should have the lowest height**.

If $\epsilon_2 = \epsilon_{\mathsf{app}} := \frac{2d_{\max}}{d_{\min}}$, then the *complementary condition* simply reduces to:

$$y_{ij:x_i} + y_{ji:x_j} \geqslant \frac{w_{ij} d(x_i, x_j)}{\epsilon_{\mathsf{app}}} \ . \tag{2}$$

It requires that any **two active labels should be raised proportionally to their "load"**.

Feasibility constraints *

To ensure feasibility of $\mathbf{y}$, `PD1` enforces for any $\alpha \in \mathcal{L}$:

$$y_{ij:\alpha} \leqslant w_{ij} d_{\min}/2 \quad \text{where} \quad d_{\min} = \min_{\alpha \neq \beta} d(\alpha, \beta) \tag{3}$$

says that **there is an upper bound on how much we can raise a label**.

Hence, we get the feasibility condition

$$y_{ij:\alpha} + y_{ji:\beta} \leqslant 2w_{ij} d_{\min}/2 = w_{ij} d_{\min} \leqslant w_{ij} d(\alpha, \beta) \ .$$

Moreover the algorithm **keeps the active balance variables non-negative**, that is $y_{ij:x_i} \geqslant 0$ for all $i \in \mathcal{V}$.

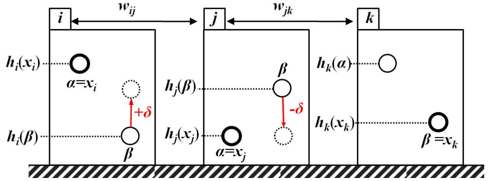
The *proportionality condition* (2) will be also fulfilled as $y_{ij:x_i}, y_{ji:x_j} \geqslant 0$ and if $y_{ij:x_i} = \frac{w_{ij} d_{\min}}{2}$, then

$$y_{ij:x_i} \geqslant \frac{w_{ij} d_{\min}}{2} \frac{d(x_i, x_j)}{d_{\max}} = \frac{w_{ij} d(x_i, x_j)}{\frac{2d_{\max}}{d_{\min}}} = \frac{w_{ij} d(x_i, x_j)}{\epsilon_{\text{app}}} \ .$$

Subroutine `Init_Primals_Duals()` *

1: $\mathbf{x}$ is simply initialized by a random label assignment ▻ Init primals
2: **for all** $(i, j) \in \mathcal{E}$ with $x_i \neq x_j$ **do** ▻ Init duals
3: $\quad y_{ij:x_i} \leftarrow w_{ij} d(x_i, x_j)/2$ and $y_{ji:x_i} \leftarrow -w_{ij} d(x_i, x_j)/2$
4: $\quad y_{ji:x_j} \leftarrow w_{ij} d(x_i, x_j)/2$ and $y_{ij:x_j} \leftarrow -w_{ij} d(x_i, x_j)/2$
5: **end for**
6: **for all** $i \in \mathcal{V}$ **do**
7: $\quad y_i \leftarrow \min_{\alpha \in \mathcal{L}} h_i(\alpha)$
8: **end for**
9: **return** $[\mathbf{x}\ \mathbf{y}]$

Update primal and dual variables



Dual variables update: Given the current active labels, any non-active label is raised, until it either reaches the active label, or attains the maximum raise allowed by the upper bound defined in (3).

Primal variables update: Given the new heights, there might still be vertices whose active labels are not at the lowest height. For each such vertex i, we select a non-active label, which is below x_i, but has already reached the maximum raise allowed by the upper bound defined in (3).

The optimal update of the α-heights can be simulated by pushing the **maximum amount of flow** through a directed graph $G' = (\mathcal{V} \cup \{s,t\}, \mathcal{E}', c, s, t)$.

Flow construction: n-links

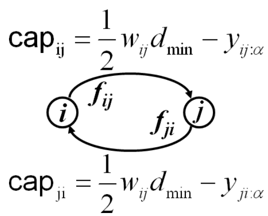
For each $(i,j) \in \mathcal{E}$, we insert two directed edges (i,j) and (j,i) into $\mathcal{E}'$.

The flow value f_{ij}, f_{ij} represent respectively the **increase, decrease of balance variable** $y_{ij:\alpha}$:

$$y'_{ij:\alpha} = y_{ij:\alpha} + f_{ij} - f_{ji} \quad \text{and} \quad y'_{ji:\alpha} = -y'_{ij:\alpha} \; .$$

According to (3), the capacities cap_{ij} and cap_{ji} are set based on

$$\text{cap}_{ij} + y_{ij:\alpha} = \frac{1}{2} w_{ij} d_{\min} = \text{cap}_{ji} + y_{ji:\alpha} \; .$$

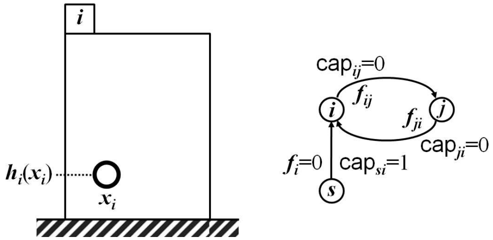


Flow construction: n-links

If α is already the active label of i (or j), then label α at i (or j) need not move.

Therefore, $y'_{ij:\alpha} = y_{ij:\alpha}$ and $y'_{ji:\alpha} = y_{ji:\alpha}$, that is

$$x_i = \alpha \text{ or } x_j = \alpha \quad \Rightarrow \quad \mathsf{cap}_{ij} = \mathsf{cap}_{ji} = 0\ .$$



Flow construction: t-links *

Each node $i \in \mathcal{V}' \backslash \{s, t\}$ connects to either the source node s or the sink node t (but not to both of them).
There are three possible cases to consider:

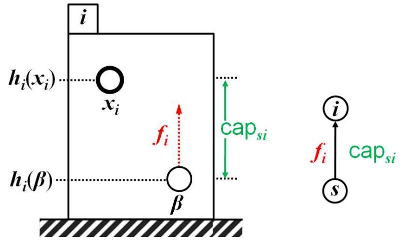
Case 1 ($h_i(\alpha) < h_i(x_i)$): we want to raise label α as much as it reaches label x_i. We connect source node s to node i.
Due to the flow conservation property, $f_i = \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} (f_{ij} - f_{ji})$ assuming the *more intuitive definition of flows* (see Lecture 4).
The flow f_i through that edge will then represent the total relative raise of label α:

$$\begin{aligned} h_i(\alpha) + f_i &= \Big(E_i(\alpha) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} (f_{ij} - f_{ji}) \\ &= \Big(E_i(\alpha) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} \big(y'_{ij:\alpha} - y_{ji:\alpha}\big) \\ &= E_i(\alpha) + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y'_{ij:\alpha} = h'_i(\alpha)\ . \end{aligned}$$

Flow construction: t-links

We need to raise up the ball corresponding to the label α only as high as the current active label of i, but not higher than that, we therefore set:

$$\text{cap}_{si} = h_i(x_i) - h_i(\alpha) \ .$$



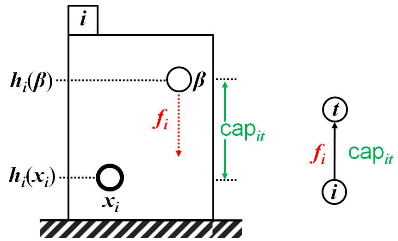
Flow construction: t-links

Case 2 ($h_i(\alpha) \geqslant h_i(x_i)$ and $c \neq x_i$): we can then afford a decrease in the height of α at i, as long as α remains above x_p.

We connect i to the sink node t through directed edge (i, t).

The flow f_i through edge it will equal the total relative decrease in the height of α:

$$h_i'(\alpha) = h_i(\alpha) - f_i$$
$$\text{cap}_{it} = h_i(\alpha) - h_i(x_i) \ .$$

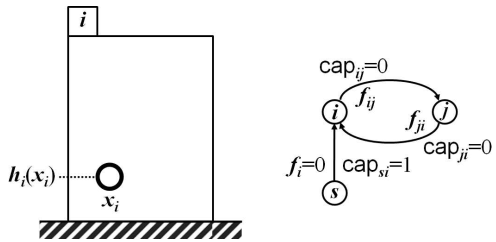


Flow construction: t-links

Case 3 ($\alpha = x_i$): we want to keep the height of α fixed at the current iteration.

Note that the capacities of the *n-edges* for p are set to 0, since i has the *active label*. Therefore, $f_i = 0$ and $h'_{ij:\alpha} = h_{ij:\alpha}$.

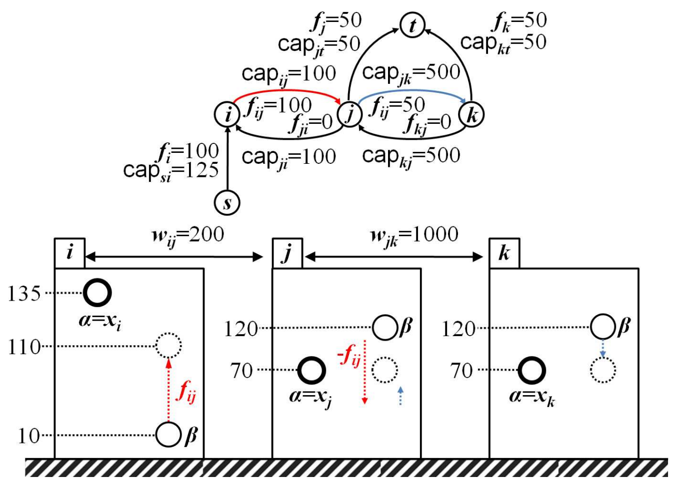
By convention cap$_{ij} := 1$.



Reassign rule

Label α will be the new label of i (i.e. $x'_i = \alpha$) iff there exists *unsaturated path* (i.e. $f_{ij} <$ cap$_{ij}$) between the source node s and node i. In all other cases, i keeps its current label (i.e. $x'_i = x_i$).

$$f_{ij} < \text{cap}_{ij}$$
$$h'_i(\alpha) - h_i(\alpha) < h_i(x_i) - h_i(\alpha)$$
$$h'_i(\alpha) < h_i(x_i) = h'_i(x_i)$$



Subroutine `Update_Duals_Primals(`α`,`$\mathbf{x}$`,`$\mathbf{y}$`)` *

$\mathbf{x}' \leftarrow \mathbf{x}$ and $\mathbf{y}' \leftarrow \mathbf{y}$
Apply max-flow to G' and compute flows f_i, f_{ij}
for all $(i,j) \in \mathcal{E}$ **do**
  $y'_{ij:\alpha} \leftarrow y_{ij:\alpha} + f_{ij} - f_{ji}$
end for
for all $i \in \mathcal{V}$ **do**
  $x_i \leftarrow \alpha \;\Leftrightarrow \exists$ unsaturated path $s \rightsquigarrow i$ in G'
end for
return $[\mathbf{x}' \; \mathbf{y}']$

Subroutine `PostEdit_Duals(`α`,`$\mathbf{x}'$`,`$\mathbf{y}'$`)` *

The goal is to restore all *active balance variables* $y_{ij:x_i}$ to be non-negative.

1. $x'_i = \alpha \neq x'_j$: we have $\text{cap}_{ij}, y_{ij:\alpha} \geqslant 0$, therefore $y'_{ij:\alpha} = \text{cap}_{ij} + y_{ij:\alpha} \geqslant 0$.
2. $x'_i = x'_j = \alpha$: we have $y'_{ij:\alpha} = -y'_{ji:\alpha}$, therefore $\text{load}'_{ij} = y'_{ij:\alpha} + y'_{ji:\alpha} = 0$. By setting $y'_{ij}(\alpha) = y'_{ji:\alpha} = 0$ we get $\text{load}'_{ij} = 0$ as well.

Note that none of the "load" were altered.

function $\text{PostEdit\_Duals}(\alpha, \mathbf{x}', \mathbf{y}')$
  for all $(i,j) \in \mathcal{E}$ with $(x'_i = x'_j = \alpha)$ and $(y'_{ij:\alpha} < 0$ or $y'_{ji:\alpha} < 0)$ **do**
    $y'_{ij:\alpha} \leftarrow 0$ and $y'_{ji:\alpha} \leftarrow 0$
  end for
  for all $i \in \mathcal{V}$ **do**
    $y'_i \leftarrow \min_{\alpha \in \mathcal{L}} h'_i(\alpha)$
  end for
  return $\mathbf{y}'$
end function

Summary *

In summary, one can see that `PD1` always leads to an ϵ-approximate solution:

Theorem 3. *The final primal-dual solutions generated by* `PD1` *satisfy*

1. $h_i(x_i) = \min_{\alpha \in \mathcal{L}} h_i(\alpha)$ *for all* $i \in \mathcal{V}$,
2. $x_i \neq x_j \Rightarrow \textit{load}_{ij} \geqslant \frac{w_{ij} d(x_i, x_j)}{\epsilon_{app}}$ *for all* $(i, j \in \mathcal{E})$,
3. $y_{ij:\alpha} \leqslant \frac{w_{ij} d_{\min}}{2}$ *for all* $(i, j \in \mathcal{E})$ *and* $\alpha \in \mathcal{L}$,

and thus they satisfy the relaxed complementary slackness conditions with $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_{app} = \frac{2 d_{\max}}{d_{\min}}$.

In the **next lecture** we will learn about

- FastPD: `PD2` and `PD3` algorithms
- `Branch-and-mincut algorithm` to achieve *global optimal* binary segmentation in case of prior information

Literature *

1. Nikos Komodakis and Georgios Tziritas. Approximate labeling via the primal-dual schema. Technical report, University of Crete, February 2005
2. Nikos Komodakis and Georgios Tziritas. Approximate labeling via graph-cuts based on linear programming. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(8):1436–1453, August 2007