Probabilistic Graphical Models in Computer Vision (IN2329)

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Recall: Inference

Inference means the procedure to estimate the *probability distribution*, encoded by a *graphical model*, for a *given data* (or observation).

Assume we are given a factor graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ and the observation $\mathbf{x}$.

- **Maximum A Posteriori (MAP) inference**: find the *state* $\mathbf{y}^* \in \mathcal{Y}$ *of maximum probability*,

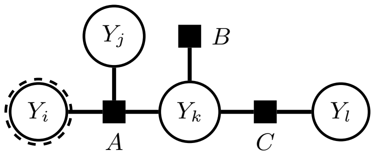
$$\mathbf{y}^* \in \operatorname*{argmax}_{\mathbf{y} \in \mathcal{Y}} p(\mathbf{y} \mid \mathbf{x}) = \operatorname*{argmin}_{\mathbf{y} \in \mathcal{Y}} E(\mathbf{y}; \mathbf{x}) \ .$$

- **Probabilistic inference**: find the value of the *partition function* $Z(\mathbf{x})$ and the *marginal distributions* $\mu_F(\mathbf{y}_F) \in \mathcal{Y}_F$ for each factor $F \in \mathcal{F}$,

$$Z(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{Y}} \exp(-E(\mathbf{y}; \mathbf{x})) \ ,$$
$$\mu_F(\mathbf{y}_F) = p(\mathbf{y}_F \mid \mathbf{x}) \ .$$

Agenda for today's lecture *

Today we are going to learn about **belief propagation** to perform **exact** inference on graphical models having **tree structure**.

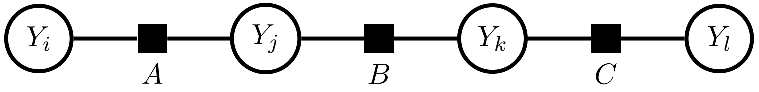


- Probabilistic inference: `Sum-product algorithm`
- MAP inference: `Max-sum algorithm`

We also extend belief propagation for **general** factor graph, which results in an **approximate** inference.

Probabilistic inference on chains

Assume that we are given the following factor graph and a corresponding energy function $E(\mathbf{y})$, where $\mathcal{Y} = \mathcal{Y}_i \times \mathcal{Y}_j \times \mathcal{Y}_k \times \mathcal{Y}_l$.



We want to compute $p(\mathbf{y})$ for any $\mathbf{y} \in \mathcal{Y}$ by making use of the factorization

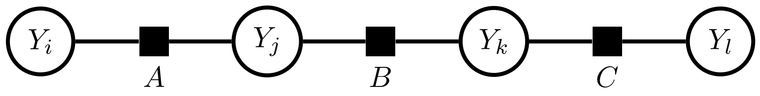
$$p(\mathbf{y}) = \frac{1}{Z}\exp(-E(\mathbf{y})) = \frac{1}{Z}\exp(-E_A(y_i, y_j))\exp(-E_B(y_j, y_k))\exp(-E_C(y_k, y_l)).$$

Problem: we also need to calculate the *partition function*

$$Z = \sum_{\mathbf{y}\in\mathcal{Y}} \exp(-E(\mathbf{y})) = \sum_{y_i\in\mathcal{Y}_i}\sum_{y_j\in\mathcal{Y}_j}\sum_{y_k\in\mathcal{Y}_k}\sum_{y_l\in\mathcal{Y}_l} \exp(-E(y_i, y_j, y_k, y_l)) ,$$

which looks expensive (the sum has $|\mathcal{Y}_i| \cdot |\mathcal{Y}_j| \cdot |\mathcal{Y}_k| \cdot |\mathcal{Y}_l|$ terms).

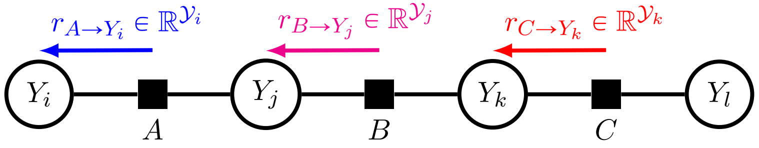
Partition function



We can expand the *partition function* as

$$
\begin{aligned}
Z &= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E(y_i, y_j, y_k, y_l)) \\
&= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp\Big(- \big(E_A(y_i, y_j) + E_B(y_j, y_k) + E_C(y_k, y_l) \big) \Big) \\
&= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \sum_{y_k \in \mathcal{Y}_k} \sum_{y_l \in \mathcal{Y}_l} \exp(-E_A(y_i, y_j)) \exp(-E_B(y_j, y_k)) \exp(-E_C(y_k, y_l)) \\
&= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) \sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_k, y_l)) \; .
\end{aligned}
$$

Elimination



Note that we can successively *eliminate* variables, that is

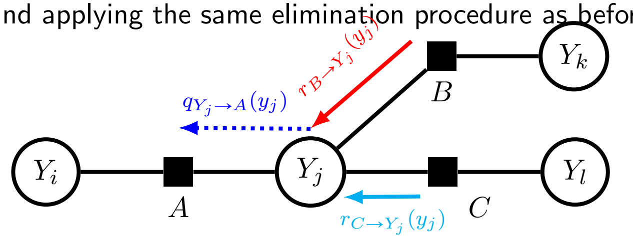
$$Z = \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) \underbrace{\sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_k, y_l))}_{r_{C\to Y_k}(y_k)}$$

$$= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \underbrace{\sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k)) r_{C\to Y_k}(y_k)}_{r_{B\to Y_j}(y_j)}$$

$$= \sum_{y_i \in \mathcal{Y}_i} \underbrace{\sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) r_{B\to Y_j}(y_j)}_{r_{A\to Y_i}(y_i)} = \sum_{y_i \in \mathcal{Y}_i} r_{A\to Y_i}(y_i) \ .$$

Inference on trees

Now we are assuming a tree-structured factor graph and applying the same elimination procedure as before.



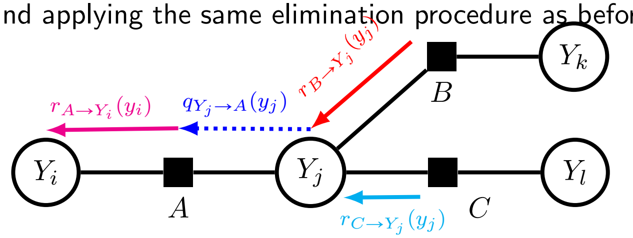
$$Z = \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \underbrace{\sum_{y_k \in \mathcal{Y}_k} \exp(-E_B(y_j, y_k))}_{r_{B\to Y_j}(y_j)} \underbrace{\sum_{y_l \in \mathcal{Y}_l} \exp(-E_C(y_j, y_l))}_{r_{C\to Y_j}(y_j)}$$

$$= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) \underbrace{r_{B\to Y_j}(y_j) r_{C\to Y_j}(y_j)}_{q_{Y_j\to A}(y_j)}$$

$$= \sum_{y_i \in \mathcal{Y}_i} \sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) q_{Y_j\to A}(y_j)$$

Inference on trees (cont.)

Now we are assuming a tree-structured factor graph and applying the same elimination procedure as before.

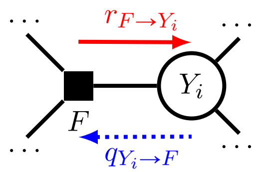


$$Z = \sum_{y_i \in \mathcal{Y}_i} \underbrace{\sum_{y_j \in \mathcal{Y}_j} \exp(-E_A(y_i, y_j)) q_{Y_j \to A}(y_j)}_{r_{A \to Y_i}(y_i)}$$

$$= \sum_{y_i \in \mathcal{Y}_i} r_{A \to Y_i}(y_i) \,.$$

Messages

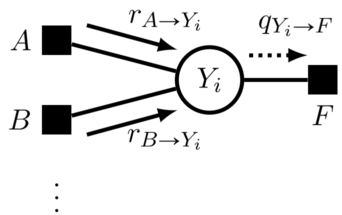
Message: pair of vectors at each factor graph edge $(i, F) \in \mathcal{E}$.



1. **Variable-to-factor** message $q_{Y_i \to F} \in \mathbb{R}^{\mathcal{Y}_i}$ is given by

$$q_{Y_i \to F}(y_i) = \prod_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y_i) \ ,$$

where $M(i) = \{F \in \mathcal{F} : (i, F) \in \mathcal{E}\}$ denotes the set of factors adjacent to Y_i.



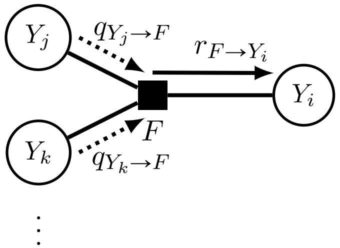
2. **Factor-to-variable** message: $r_{F \to Y_i} \in \mathbb{R}^{\mathcal{Y}_i}$.

Factor-to-variable message

2. **Factor-to-variable** message $r_{F\to Y_i} \in \mathbb{R}^{\mathcal{Y}_i}$ is given by

$$r_{F\to Y_i}(y_i) = \sum_{\substack{\mathbf{y}'_F \in \mathcal{Y}_F,\\ y'_i = y_i}} \left(\exp(-E_F(\mathbf{y}'_F)) \prod_{l \in N(F)\setminus\{i\}} q_{Y_l \to F}(y'_l) \right),$$

where $N(F) = \{i \in V : (i, F) \in \mathcal{E}\}$ denotes the set of variables adjacent to F.



Message scheduling *

One can note that the message updates depend on each other.

$$r_{F\to Y_i}(y_i) = \sum_{\substack{\mathbf{y}'_F \in \mathcal{Y}_F,\\ y'_i = y_i}} \left(\exp(-E_F(\mathbf{y}'_F)) \prod_{l \in N(F)\setminus\{i\}} q_{Y_l \to F}(y'_l) \right) \tag{1}$$

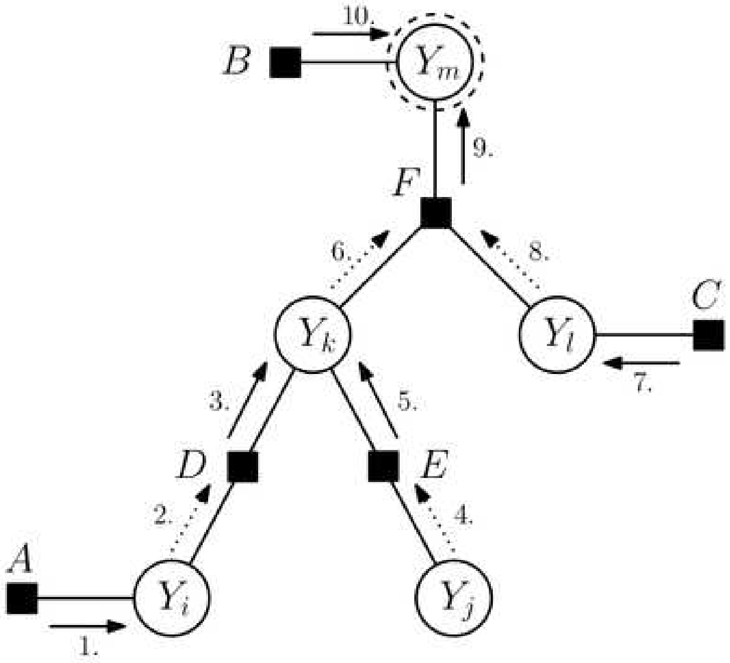
$$q_{Y_i\to F}(y_i) = \prod_{F' \in M(i)\setminus\{F\}} r_{F'\to Y_i}(y_i) \tag{2}$$

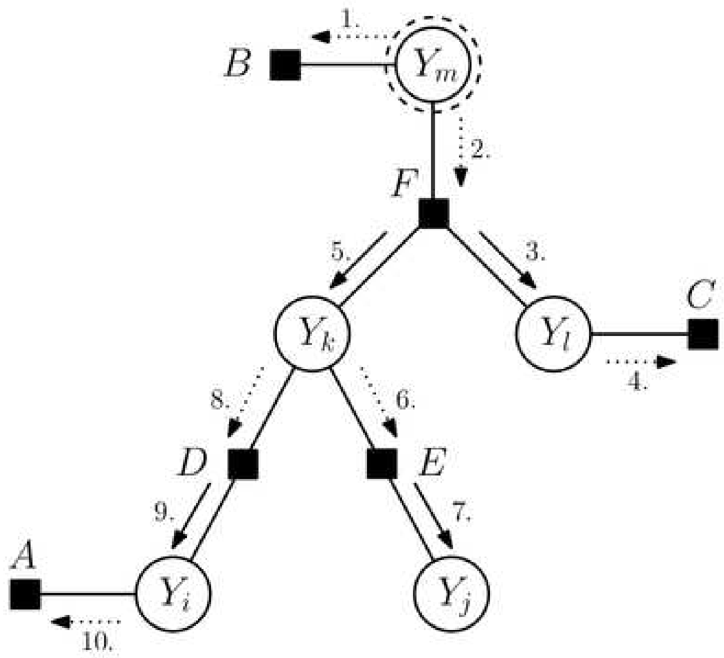
The messages that do not depend on previous computation are the following.

- The factor-to-variable messages in which no other variable is adjacent to the factor; then the product in (1) will be empty.
- The variable-to-factor messages in which no other factor is adjacent to the variable; then the product in (2) is empty and the message will be one.

Message scheduling on trees

For tree-structured factor graphs there always exist at least one such message that can be computed initially, hence all the dependencies can be resolved.



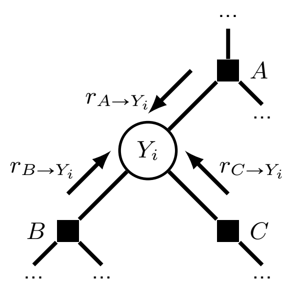


1. Select one variable node as root of the tree (e.g., Y_m)
2. Compute leaf-to-root messages (e.g., by applying depth-first-search)
3. Compute root-to-leaf messages (reverse order as before)

Inference result: partition function Z

Partition function is evaluated at the (root) node i

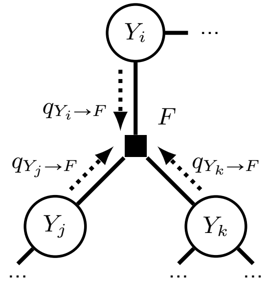
$$Z = \sum_{y_i \in \mathcal{Y}_i} \prod_{F \in M(i)} r_{F \to Y_i}(y_i) \ .$$



Inference result: the marginals $\mu_F(\mathbf{y}_F)$

The marginal distribution for each factor can be computed as

$$\begin{aligned}\mu_F(\mathbf{y}_F) = \sum_{\substack{\mathbf{y}'\in\mathcal{Y},\\ \mathbf{y}'_F=\mathbf{y}_F}} p(\mathbf{y}) = \sum_{\substack{\mathbf{y}'\in\mathcal{Y},\\ \mathbf{y}'_F=\mathbf{y}_F}} \frac{1}{Z}\exp(-\sum_{H\in\mathcal{F}} E_H(\mathbf{y}'_H))\\ =\frac{1}{Z}\exp(-E_F(\mathbf{y}_F)) \sum_{\mathbf{y}'\in \times_{H\in\mathcal{F}\setminus\{F\}} \mathcal{Y}_H} \exp(\sum_{H\in\mathcal{F}\setminus\{F\}} -E_H(\mathbf{y}'_H))\\ =\frac{1}{Z}\exp(-E_F(\mathbf{y}_F)) \prod_{i\in N(F)} q_{Y_i\to F}(y_i)\ .\end{aligned}$$



Optimality and complexity *

Assume a *tree-structured* factor graph. If the messages are computed based on *depth-first search order* for the *sum-product algorithm*, then it converges after $2|V|$ iterations and provides the **exact** marginals.

If $|\mathcal{Y}_i| \leqslant m$ for all $i \in V$, then the complexity of the algorithm $\mathcal{O}(|V| \cdot m^K)$, where $K = \max_{F\in\mathcal{F}} |N(F)|$.

$$r_{F\to Y_i}(y_i) = \sum_{\substack{\mathbf{y}'_F\in\mathcal{Y}_F,\\ y'_i=y_i}} \left(\exp(-E_F(\mathbf{y}'_F)) \prod_{l\in N(F)\setminus\{i\}} q_{Y_l\to F}(y'_l) \right) .$$

Note that the complexity of the naïve way is $\mathcal{O}(K \cdot m^{|V|})$.

Reminder: Assuming $f, g : \mathbb{R} \to \mathbb{R}$, the notation $f(x) = \mathcal{O}(g(x))$ means that there exists $C > 0$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leqslant C|g(x)|$ for all $x > x_0$.

MAP inference

$$\mathbf{y}^* \in \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} p(\mathbf{y}) = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \frac{1}{Z}\tilde{p}(\mathbf{y}) = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \tilde{p}(\mathbf{y}) \; .$$

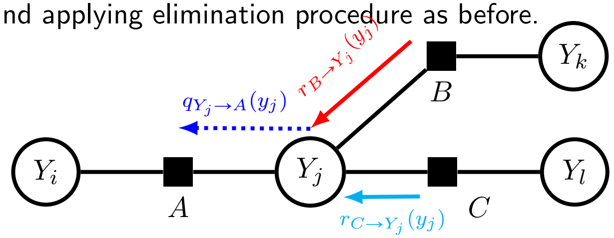
Similar to the *sum-product algorithm* one can obtain the so-called **max-sum algorithm** to solve the above maximization.

By applying the $\ln$ function, we have

$$\begin{aligned} \ln \max_{\mathbf{y}\in\mathcal{Y}} \tilde{p}(\mathbf{y}) &= \max_{\mathbf{y}\in\mathcal{Y}} \ln \tilde{p}(\mathbf{y}) \\ &= \max_{\mathbf{y}\in\mathcal{Y}} \ln \prod_{F\in\mathcal{F}} \exp(-E_F(\mathbf{y}_F)) \\ &= \max_{\mathbf{y}\in\mathcal{Y}} \sum_{F\in\mathcal{F}} -E_F(\mathbf{y}_F) \; . \end{aligned}$$

MAP inference on trees

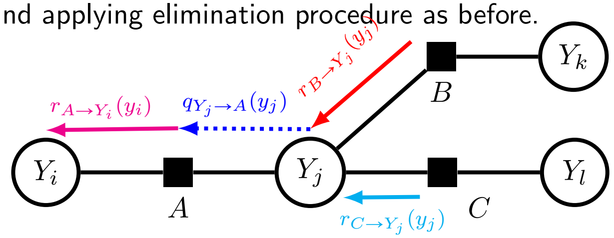
Now we are assuming a tree-structured factor graph and applying elimination procedure as before.



$$\begin{aligned}
\max_{\mathbf{y}\in\mathcal{Y}} \sum_{F\in\mathcal{F}} -E_F(\mathbf{y}_F) &= \max_{\mathbf{y}} -E_A(y_i,y_j) - E_B(y_j,y_k) - E_C(y_j,y_l) \\
&= \max_{y_i,y_j} -E_A(y_i,y_j) + \underbrace{\max_{y_k} -E_B(y_j,y_k)}_{r_{B\to Y_j}(y_j)} + \underbrace{\max_{y_l} -E_C(y_j,y_l)}_{r_{C\to Y_j}(y_j)} \\
&= \max_{y_i,y_j} -E_A(y_i,y_j) + \underbrace{r_{B\to Y_j}(y_j) + r_{C\to Y_j}(y_j)}_{q_{Y_j\to A}(y_j)}
\end{aligned}$$

MAP inference on trees (cont.)

Now we are assuming a tree-structured factor graph and applying elimination procedure as before.



$$\max_{\mathbf{y} \in \mathcal{Y}} \sum_{F \in \mathcal{F}} -E_F(\mathbf{y}_F) = \max_{y_i} \underbrace{\max_{y_j} -E_A(y_i, y_j) + q_{Y_j \to A}(y_j)}_{r_{A \to Y_i}(y_i)} = \max_{y_i} r_{A \to Y_i}(y_i)$$

The solution is then obtained as:

$$y_i^* \in \operatorname*{argmax}_{y_i} r_{A \to Y_i}(y_i), \qquad y_j^* \in \operatorname*{argmax}_{y_j} E_A(y_i^*, y_j) + q_{Y_j \to A}(y_j),$$

$$y_k^* \in \operatorname*{argmax}_{y_k} E_B(y_j^*, y_k), \qquad y_l^* \in \operatorname*{argmax}_{y_l} E_C(y_j^*, y_l)$$

Messages

The messages become as follows

$$q_{Y_i \to F}(y_i) = \sum_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y_i)$$

$$r_{F \to Y_i}(y_i) = \max_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(-E_F(y'_F) + \sum_{l \in N(F) \setminus \{i\}} q_{Y_l \to F}(y'_l) \right).$$

The max-sum algorithm provides **exact** MAP inference for tree-structured factor graphs.

In general, for graphs with cycles there is no guarantee for convergence.

Choosing an optimal state *

The following **back-tracking** algorithm is applied for choosing an optimal y^*.

1. Initialize the procedure at the root node (Y_i) by choosing any

$$y_i^* \in \operatorname*{argmax}_{y_i \in \mathcal{Y}_i} \max_{\mathbf{y}' \in \mathcal{Y}, y_i' = y_i} \tilde{p}(\mathbf{y}') \ ,$$

   and set $\mathcal{I} = \{i\}$.

2. Based on *(reverse) depth-first search order*, for each $j \in \mathcal{V} \backslash \mathcal{I}$
   (a) choose a configuration y_j^* at the node Y_j such that

$$y_j^* \in \operatorname*{argmax}_{y_j \in \mathcal{Y}_j} \max_{\substack{\mathbf{y}' \in \mathcal{Y}, \\ y_j' = y_j, \\ y_i' = y_i^* \ \forall i \in \mathcal{I}}} \tilde{p}(\mathbf{y}') \ ,$$

   (b) update $\mathcal{I} = \mathcal{I} \cup \{j\}$.

Sum-product and Max-sum comparison *

- Sum-product algorithm

$$q_{Y_i \to F}(y_i) = \prod_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y_i)$$

$$r_{F \to Y_i}(y_i) = \sum_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(\exp(-E_F(y'_F)) \prod_{l \in N(F) \setminus \{i\}} q_{Y_l \to F}(y'_l) \right)$$

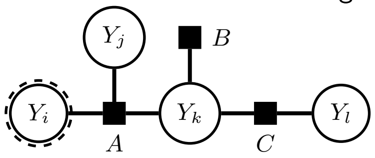
- Max-sum algorithm

$$q_{Y_i \to F}(y_i) = \sum_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y_i)$$

$$r_{F \to Y_i}(y_i) = \max_{\substack{y'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(-E_F(y'_F) + \sum_{l \in N(F) \setminus \{i\}} q_{Y_l \to F}(y'_l) \right)$$

Example $*$

Let us consider the following factor graph with binary variables:



$E_A(0, y_j, y_k)$

		y_k 0	y_k 1
y_j	0	1	0
y_j	1	0	1

$E_A(1, y_j, y_k)$

		y_k 0	y_k 1
y_j	0	0	-1
y_j	1	0	0

$E_B(y_k)$

	y_k
0	1
1	0.5

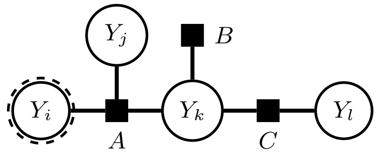
$E_C(y_k, y_l)$

		y_l 0	y_l 1
y_k	0	0	0.5
y_k	1	0.5	0

Let us chose the node Y_i as root. We calculate the messages for the max-sum algorithm from leaf–to–root direction in a topological order as follows.

1. $q_{Y_l \to C}(0) = q_{Y_l \to C}(1) = 0$
2. $r_{C \to Y_k}(0) = \max_{y_l \in \{0,1\}} \{-E_C(0, y_l) + q_{Y_l \to C}(0)\} = \max_{y_l \in \{0,1\}} -E_C(0, y_l) = 0$
   $r_{C \to Y_k}(1) = \max_{y_l \in \{0,1\}} \{-E_C(1, y_l) + q_{Y_l \to C}(1)\} = \max_{y_l \in \{0,1\}} -E_C(1, y_l) = 0$
3. $r_{B \to Y_k}(0) = -1$
   $r_{B \to Y_k}(1) = -0.5$
4. $q_{Y_k \to A}(0) = r_{B \to Y_k}(0) + r_{C \to Y_k}(0) = -1 + 0 = -1$
   $q_{Y_k \to A}(1) = r_{B \to Y_k}(1) + r_{C \to Y_k}(1) = -0.5 + 0 = -0.5$

Example (cont.) *



5. $q_{Y_j\to A}(0) = q_{Y_j\to A}(1) = 0$
6. $r_{A\to Y_i}(0) = \max_{y_j,y_k\in\{0,1\}}\{-E_A(0,y_j,y_k)+q_{Y_j\to A}(y_j)+q_{Y_k\to A}(y_k)\} = -0.5$
   $r_{A\to Y_i}(1) = \max_{y_j,y_k\in\{0,1\}}\{-E_A(1,y_j,y_k)+q_{Y_j\to A}(y_j)+q_{Y_k\to A}(y_k)\} = 0.5$

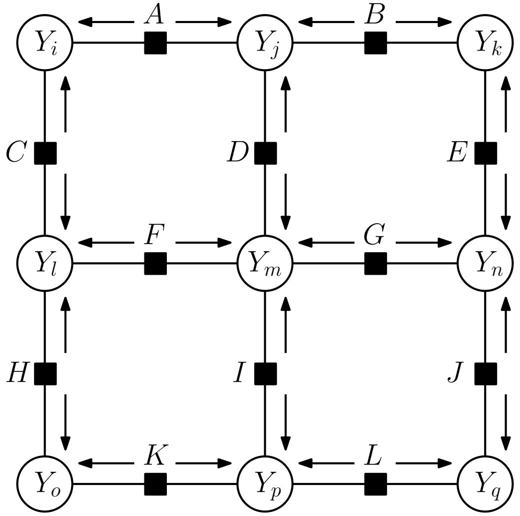
In order to calculate the maximal state y^* we apply *back-tracking*

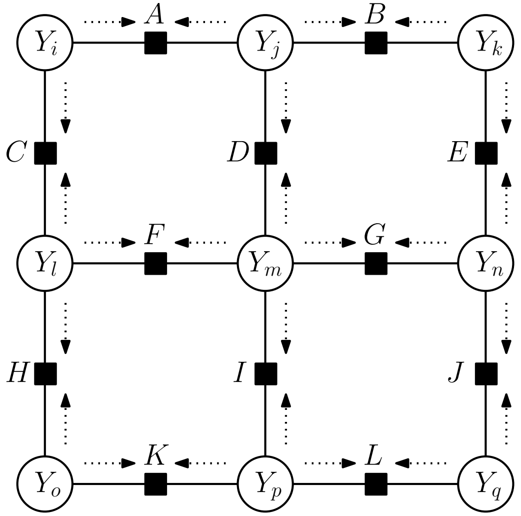
1. $y_i^* \in \operatorname{argmax}_{y_i\in\{0,1\}} r_{A\to Y_i}(y_i) = \{1\}$
2. $y_j^* \in \operatorname{argmax}_{y_j} \max_{y_j,y_k\in\{0,1\}}\{-E_A(1,y_j,y_k) + q_{Y_k\to A}(y_k)\} = \{0\}$
3. $y_k^* \in \operatorname{argmax}_{y_k\in\{0,1\}}\{-E_A(1,0,y_k) + r_{B\to Y_k}(y_k) + r_{C\to Y_k}(y_k)\} = \{1\}$
4. $y_l^* \in \operatorname{argmax}_{y_l\in\{0,1\}}\{-E_C(1,y_l) + r_{C\to Y_k}(1)\} = \{1\}$

Therefore, the optimal state $y^* = (y_i^*, y_j^*, y_k^*, y_l^*) = (1,0,1,1)$.

Message passing in cyclic graphs

When the graph has cycles, then there is no well-defined *leaf–to–root* order. However, one can apply message passing on cyclic graphs, which results in **loopy belief propagation**.





1. Initialize all messages as constant 1
2. Pass factor–to–variables and variables–to–factor messages alternately until convergence
3. Upon convergence, treat **beliefs** μ_F as approximate marginals

Messages

The **factor–to–variable messages** $r_{F \to Y_i}$ remain well-defined and are computed as before.

$$r_{F \to Y_i}(y_i) = \sum_{\substack{\mathbf{y}'_F \in \mathcal{Y}_F, \\ y'_i = y_i}} \left(\exp(-E_F(\mathbf{y}'_F)) \prod_{j \in N(F) \setminus \{i\}} q_{Y_j \to F}(y'_j) \right)$$

The **variable–to–factor messages** are normalized at every iteration as follows:

$$q_{Y_i \to F}(y_i) = \frac{\prod_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y_i)}{\sum_{y'_i \in \mathcal{Y}_i} \prod_{F' \in M(i) \setminus \{F\}} r_{F' \to Y_i}(y'_i)} \,.$$

In case of tree structured graphs, in the sum–product algorithm these normalization constants are equal to 1, since the marginal distributions, calculated in each iteration, are exact.

Beliefs

The approximate marginals, i.e.**beliefs**, are computed as before but now a factor-specific normalization constant z_F is also used.

The **factor marginals** are given by

$$\mu_F(y_F) = \frac{1}{z_F} \exp(-E_F(y_F)) \prod_{i \in N(F)} q_{Y_i \to F}(y_i) \ ,$$

where the factor specific *normalization constant* is given by

$$z_F = \sum_{y_F \in \mathcal{Y}_F} \exp(-E_F(y_F)) \prod_{i \in N(F)} q_{Y_i \to F}(y_i) \ .$$

Beliefs (cont.) *

In addition to the factor marginals the algorithm also computes the **variable marginals** in a similar fashion.

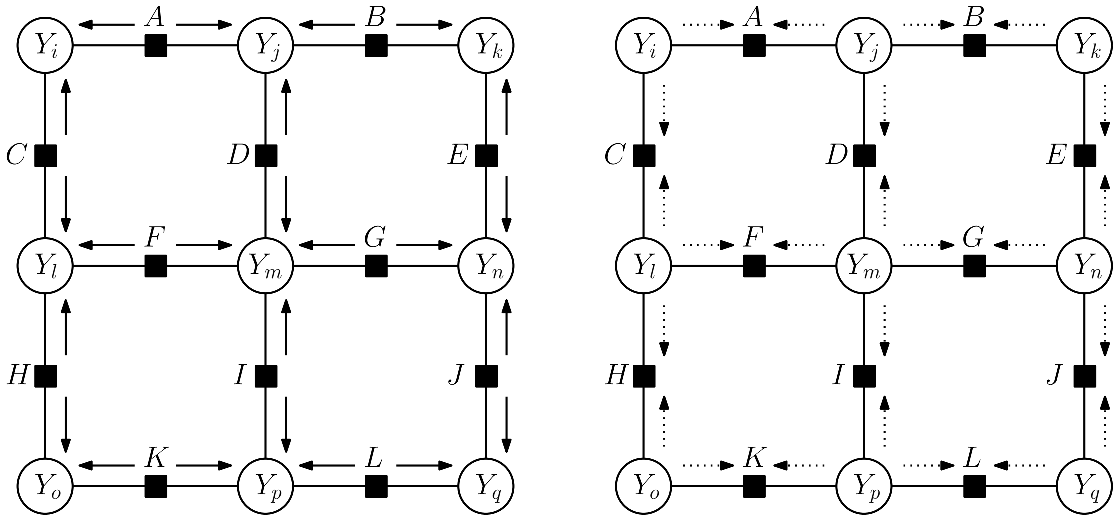
$$\mu_i(y_i) = \frac{1}{z_i} \prod_{F' \in M(i)} r_{F' \to Y_i}(y_i) \ ,$$

where the normalizing constant is given by

$$z_i = \sum_{y_i \in \mathcal{Y}_i} \prod_{F' \in M(i)} r_{F' \to Y_i}(y_i) \ .$$

Since the local *normalization constant* z_F differs at each factor for loopy belief propagation, the exact value of the normalizing constant Z **cannot** be directly calculated. Instead, an approximation to the log partition function can be computed.

Remarks on loopy belief propagation



Loopy belief propagation is very popular, but has some problems:

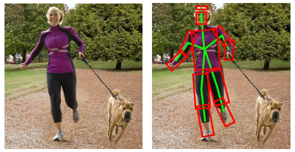
- It might not converge (e.g., it can oscillate).
- Even if it does, the computed probabilities are only *approximate*.
- If there is a single cycle only in the graph, then it converges.

Summary *

- We have discussed inference methods on *tree-structured* graphical models
  - Probabilistic inference: `Sum-product algorithm`
  - MAP inference: `Max-sum algorithm`
- For *general* factor graphs: `Loopy belief propagation`

In the **next lecture** we will learn about

- Human-pose estimation



- *Mean-field approximation*: probabilistic inference via optimization (a.k.a. variational inference)

Literature *

1. Sebastian Nowozin and Christoph H. Lampert. Structured prediction and learning in computer vision. *Foundations and Trends in Computer Graphics and Vision*, 6(3–4), 2010
2. Daphne Koller and Nir Friedman. *Probabilistic Graphical Models: Principles and Techniques*. MIT Press, 2009
3. Judea Pearl. *Probabilistic Reasoning in Intelligent Systems: Network of Plausible Inference*. Morgan Kaufmann, 1988