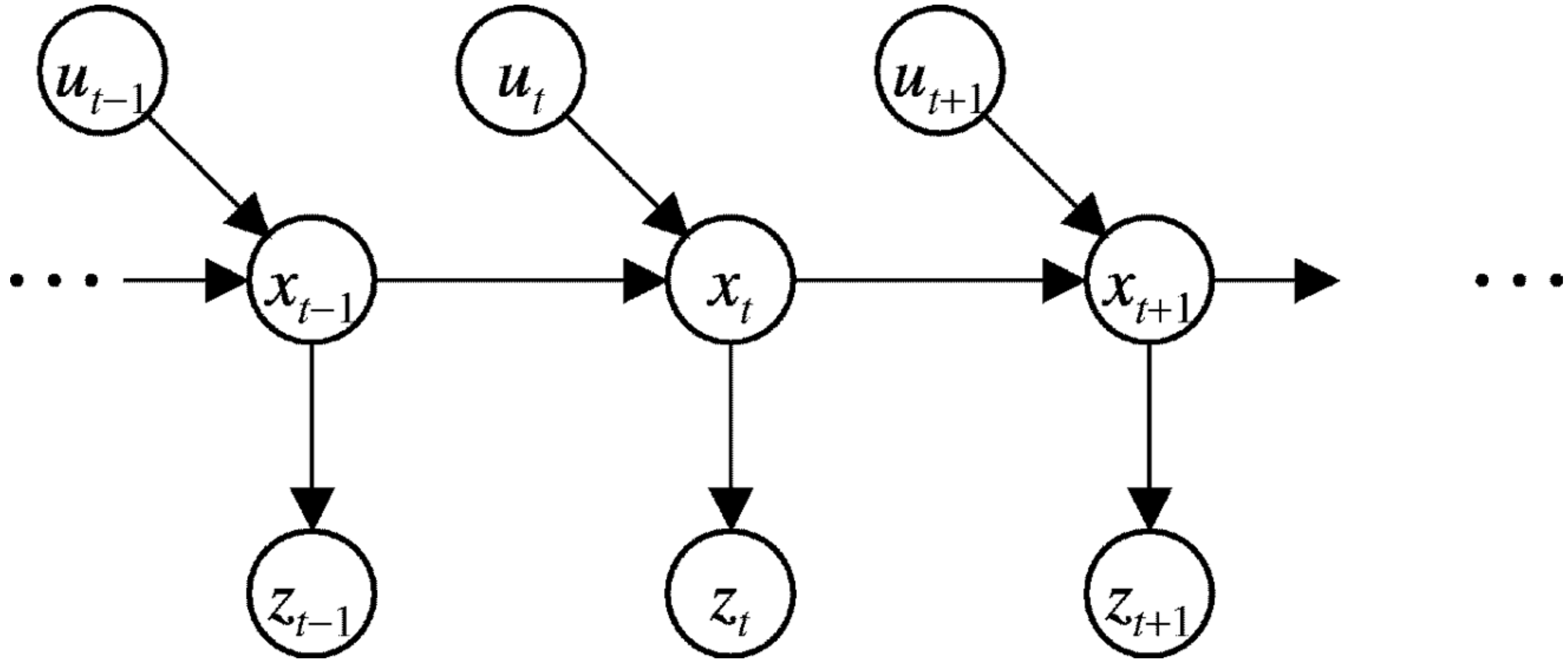
5. Hidden Markov Models

Bayes Filter (Rep.)

We can describe the overall process using a *Dynamic Bayes Network*:



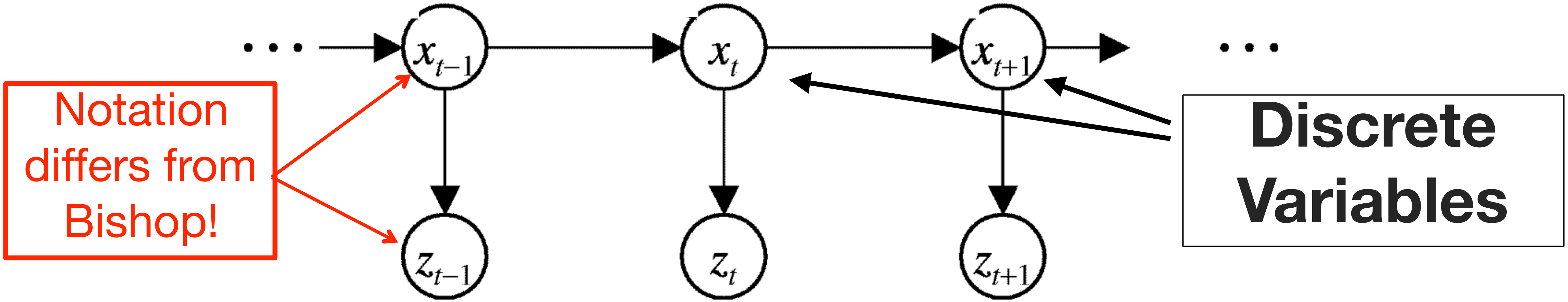
- This incorporates the following Markov assumptions:

$$p(z_t \mid x_{0:t}, u_{1:t}, z_{1:t}) = p(z_t \mid x_t) \quad \text{(measurement)}$$

$$p(x_t \mid x_{0:t-1}, u_{1:t}, z_{1:t}) = p(x_t \mid x_{t-1}, u_t) \quad \text{(state)}$$

Bayes Filter Without Actions

Removing the action variables we obtain:



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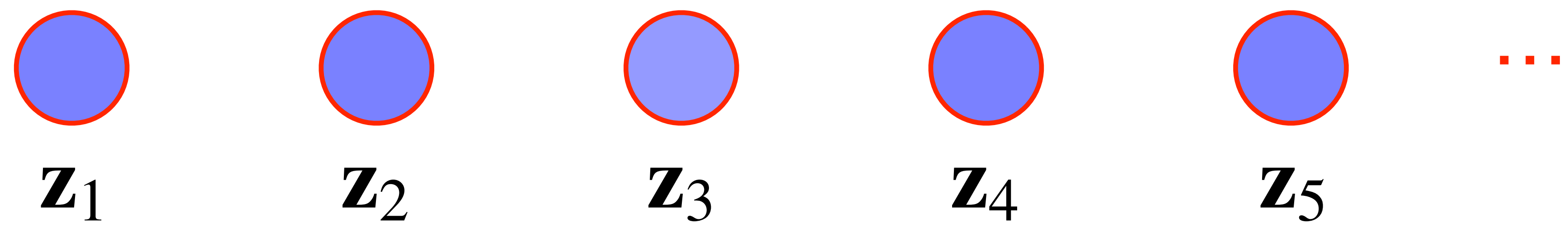
$$p(z_t \mid x_{0:t}, \quad z_{1:t}) = p(z_t \mid x_t) \quad \text{(measurement)}$$

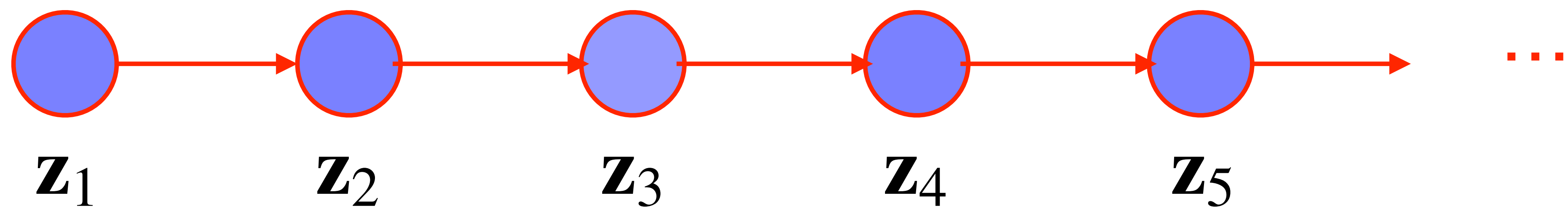
$$p(x_t \mid x_{0:t-1}, \quad z_{1:t}) = p(x_t \mid x_{t-1} \quad) \quad \text{(state)}$$

A Model for Sequential Data

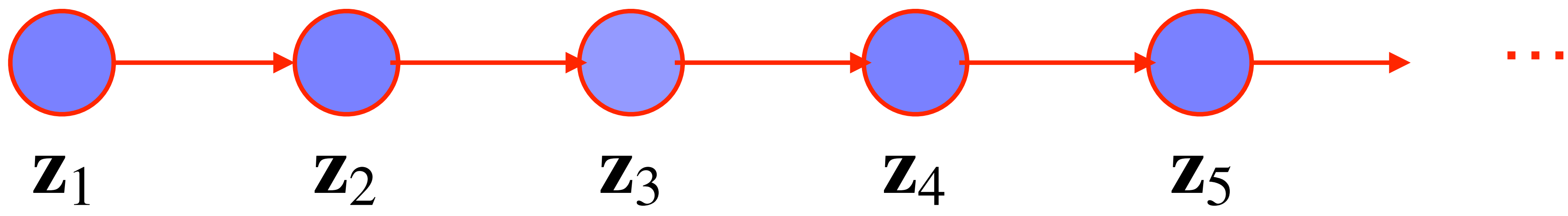
- Observations in sequential data should not be modeled as independent variables such as:



- Examples: weather forecast, speech, hand-written text, etc.
- The observation at time t depends on the observation(s) of (an) earlier time step(s):



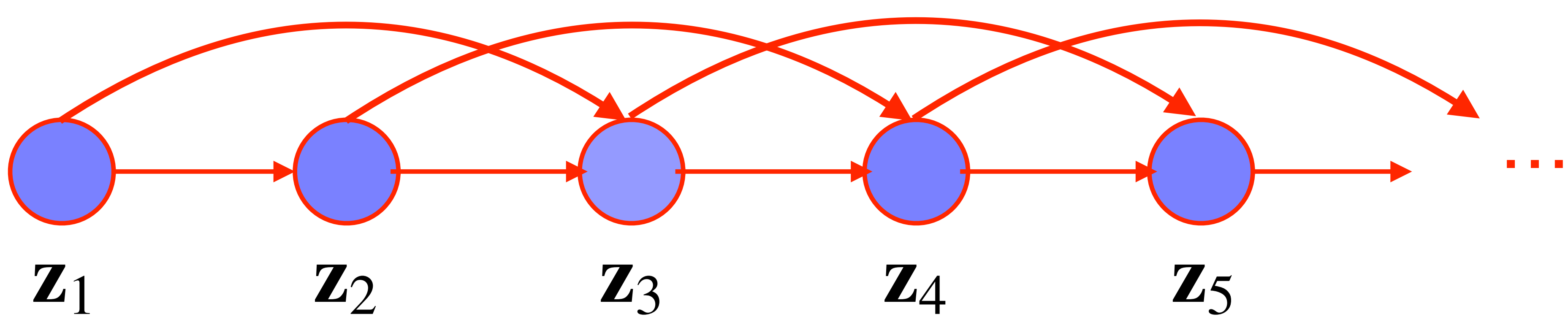
A Model for Sequential Data



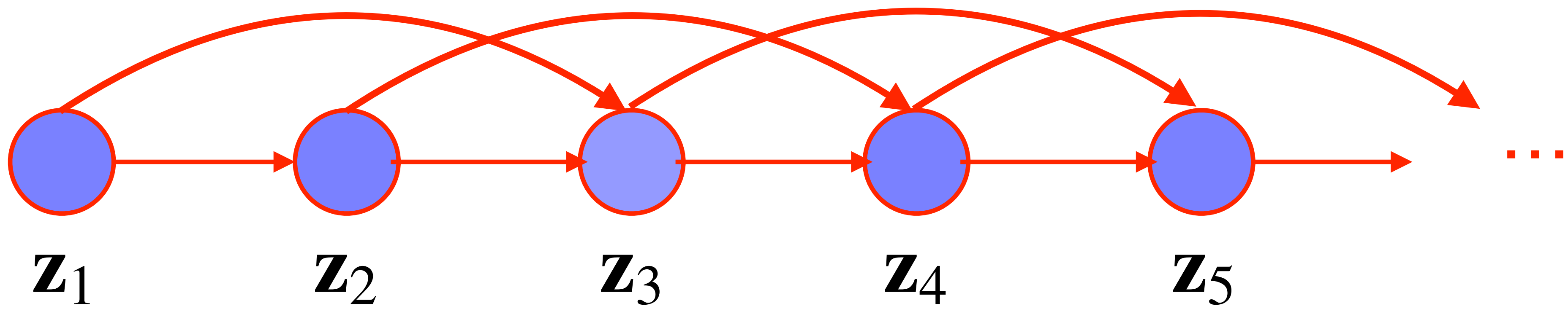
- The joint distribution is therefore (d-sep):

$$p(\mathbf{z}_1 \ldots \mathbf{z}_n) = p(\mathbf{z}_1) \prod_{i=2}^{n} p(\mathbf{z}_i \mid \mathbf{z}_{i-1})$$

- **However:** often data depends on several earlier observations (not just one)



A Model for Sequential Data

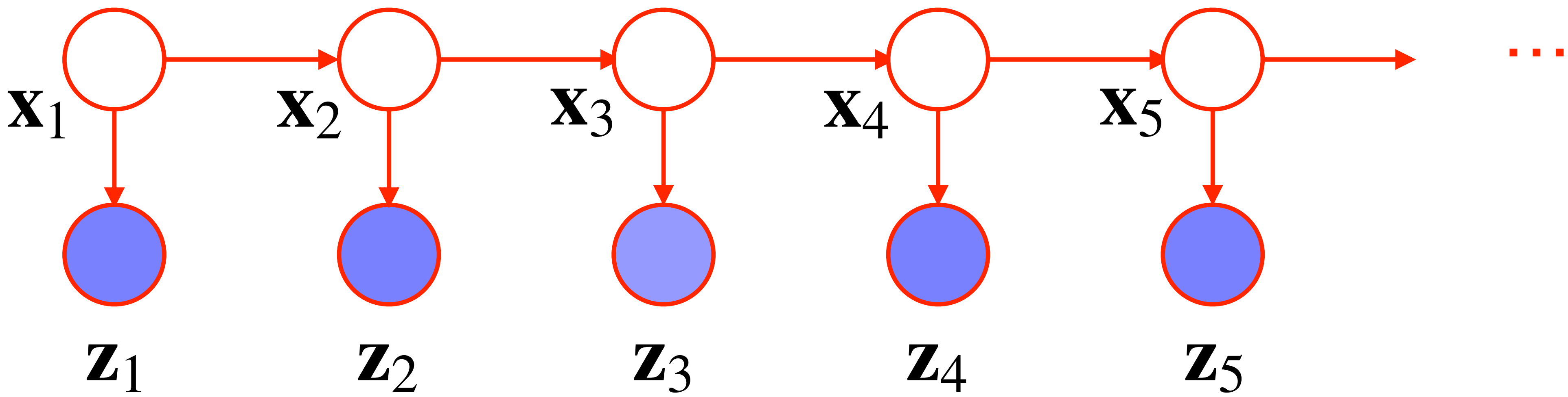


$$p(\mathbf{z}_1 \ldots \mathbf{z}_n) = p(\mathbf{z}_1)p(\mathbf{z}_2 \mid \mathbf{z}_1)\prod_{i=3}^{n} p(\mathbf{z}_i \mid \mathbf{z}_{i-1}, \mathbf{z}_{i-2})$$

- **Problem:** number of stored parameters grows exponentially with the **order** of the Markov chain
- **Question:** can we model dependency of all previous observations with a limited number of parameters?

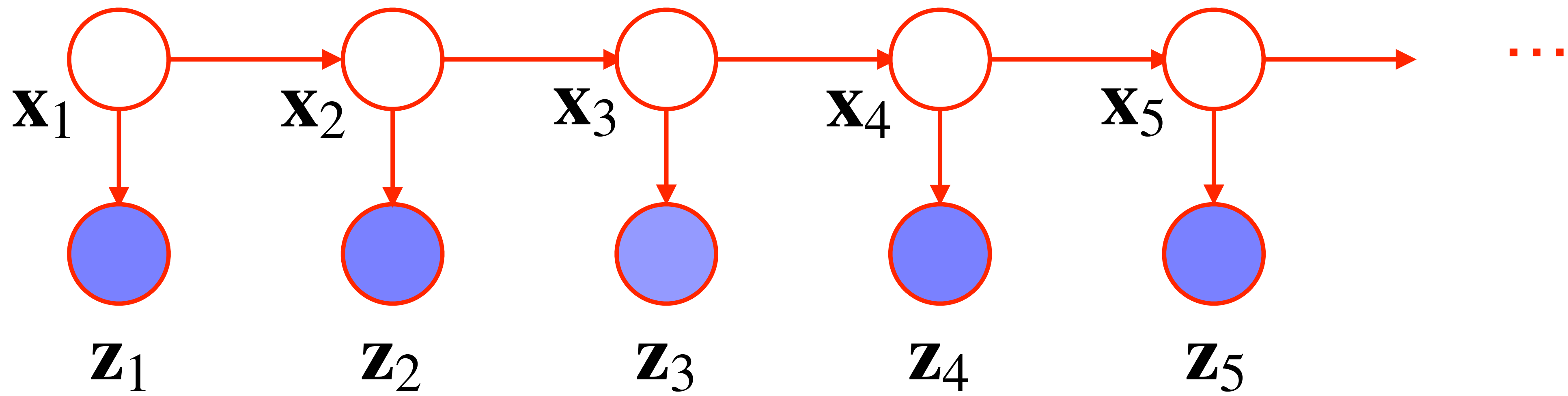
A Model for Sequential Data

Idea: Introduce **hidden** (unobserved) variables:



A Model for Sequential Data

Idea: Introduce **hidden** (unobserved) variables:



Now we have: $\mathrm{dsep}(\mathbf{x}_n, \{\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}\}, \mathbf{x}_{n-1})$

$$\Leftrightarrow p(\mathbf{x}_n \mid \mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_{n-1}) = p(\mathbf{x}_n \mid \mathbf{x}_{n-1})$$

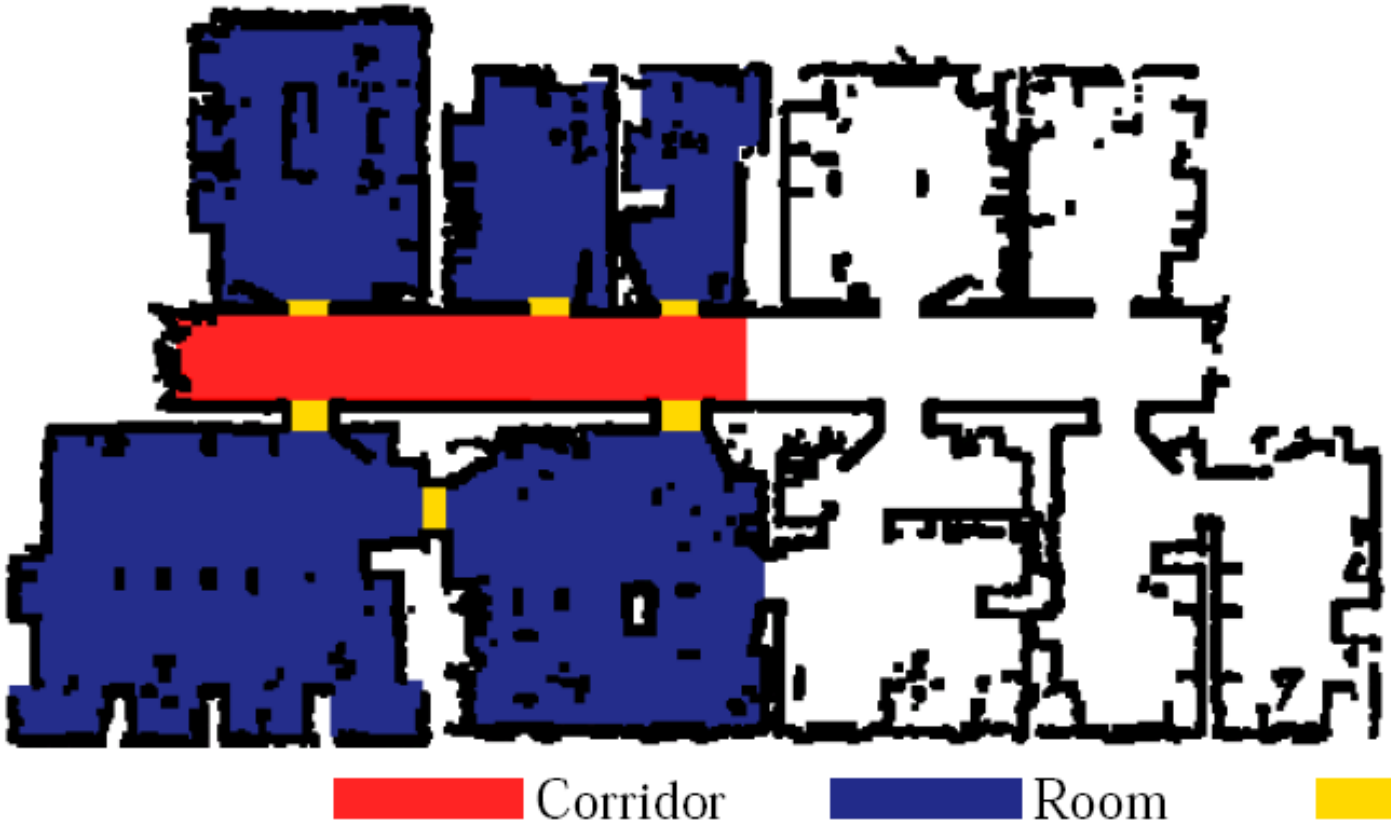
But: $\neg\mathrm{dsep}(\mathbf{z}_n, \{\mathbf{z}_1, \ldots, \mathbf{z}_{n-2}\}, \mathbf{z}_{n-1})$

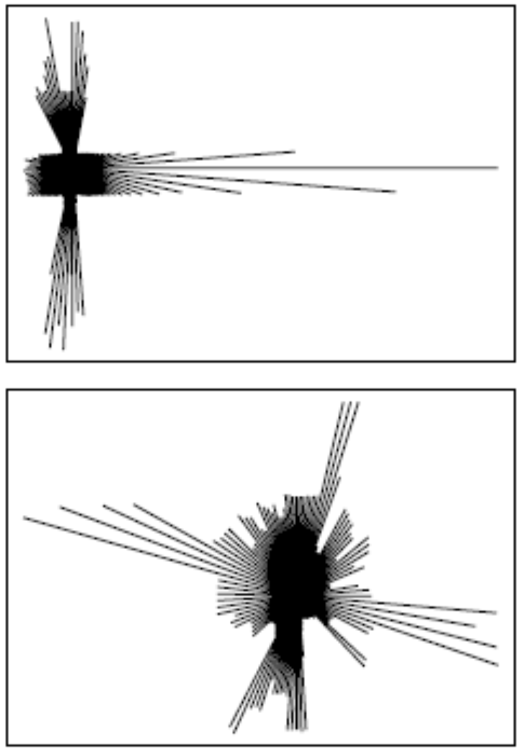
$$\Leftrightarrow p(\mathbf{z}_n \mid \mathbf{z}_1, \ldots, \mathbf{z}_{n-2}, \mathbf{z}_{n-1}) \neq p(\mathbf{z}_n \mid \mathbf{z}_{n-1})$$

And: number of parameters is $nK(K\text{-}1) + const.$

Example

- Place recognition for mobile robots
- 3 different states: corridor, room, doorway
- Problem: misclassifications
- Idea: use information from previous time step





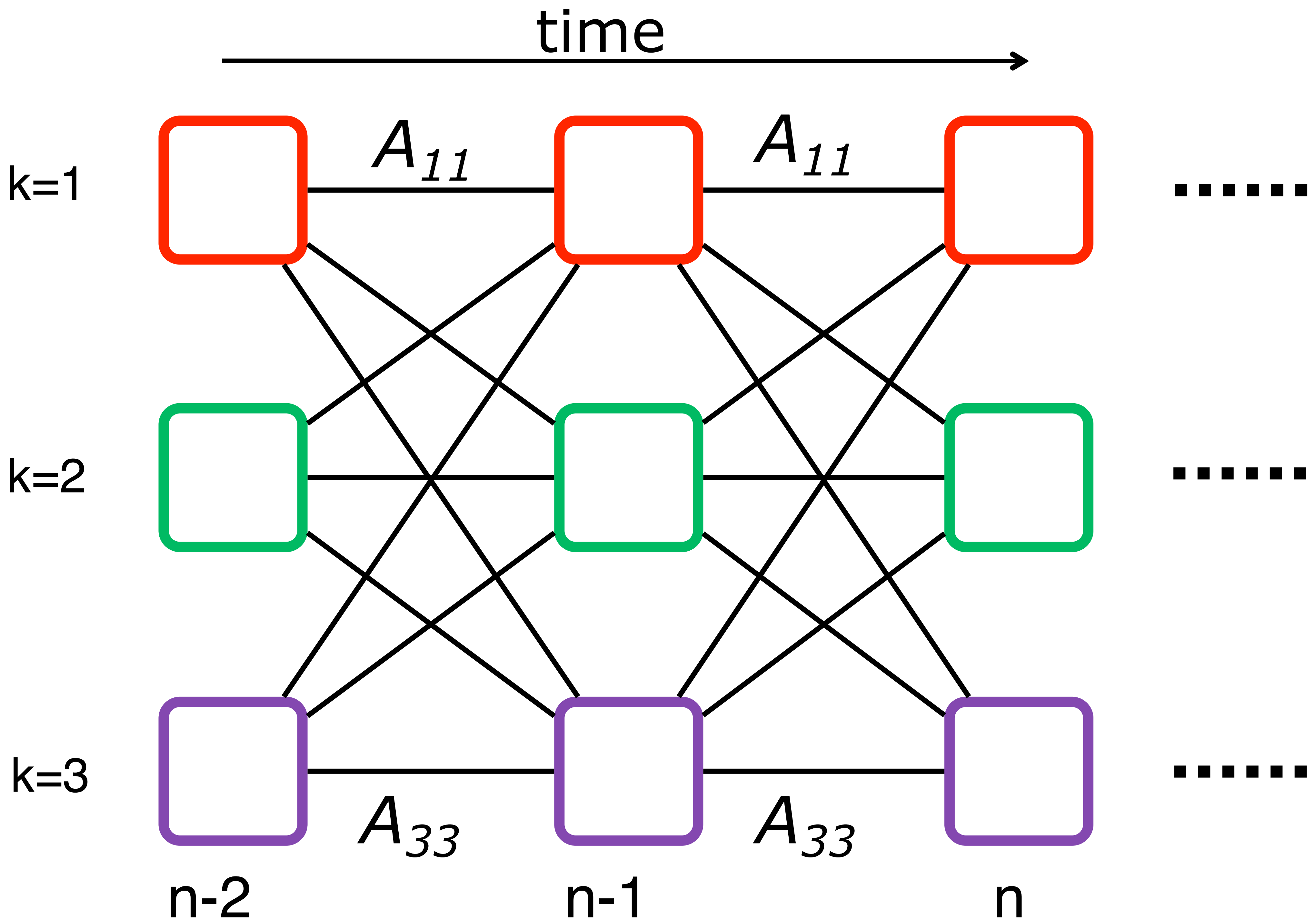
General Formulation of an HMM

1. Discrete random variables
   - Observation variables: $\{z_n\}$, n = 1..N
   - Discrete state variables (unobservable): $\{x_n\}$, n = 1..N
   - Number of states K: $x_n \epsilon \{1 \ldots K\}$
2. Transition model $p(x_i \mid x_{i-1})$
   - Markov assumption (x_i only depends on x_{i-1})
   - Represented as a $K \times K$ **transition matrix** A
   - Initial probability: $p(x_0)$ repr. as π_1, π_2, π_3
3. Observation model $p(z_i \mid x_i)$ with parameters φ
   - Observation only depends on the current state
   - Example: output of a "local" place classifier

Model Parameters

θ

The Trellis Representation



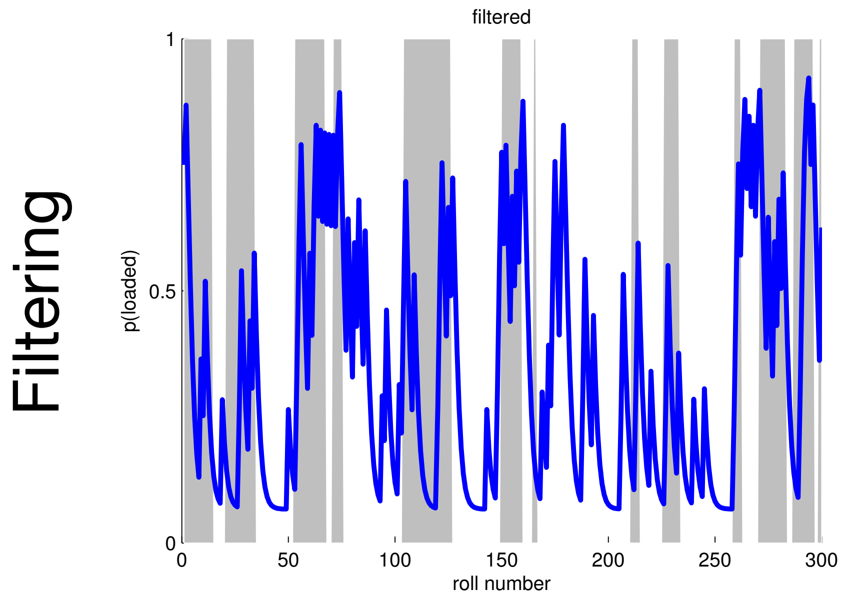
Application Example (1)

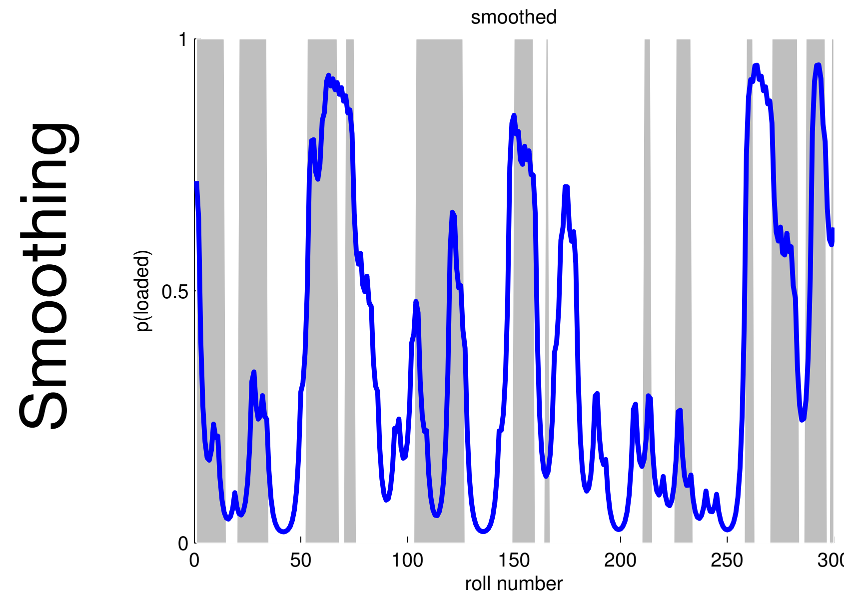
- Given an observation sequence $z_1, z_2, z_3 \ldots$
- Assume that the model parameters $\theta = (A, \pi, \varphi)$ are known
- What is the probability that the given observation sequence is actually observed under this model, i.e. the **data likelihood** $p(Z|\ \theta)$?
- If we are given several different models, we can choose the one with highest probability
- Expressed as a **supervised learning problem**, this can be interpreted as the inference step (classification step)

Application Example (2)

Based on the data likelihood we can solve two different kinds of problems:

- **Filtering:** computes $p(x_t \mid \mathbf{z}_{1:t})$, i.e. state probability only based on previous observations
- **Smoothing:** computes $p(x_t \mid \mathbf{z}_{1:T})$, state probability based on **all** observations (including those from the future)





Application Example (3)

- Given an observation sequence $z_1,z_2,z_3\ldots$
- Assume that the model parameters $\theta = (A, \pi, \varphi)$ are known
- What is the state sequence $x_1,x_2,x_3\ldots$ that explains best the given observation sequence?
- In the case of place recognition: which is the sequence of truly visited places that explains best the sequence of obtained place labels (classifications)?

Application Example (4)

- Given an observation sequence $z_1, z_2, z_3 \ldots$
- What are the optimal model parameters $\theta = (A, \pi, \varphi)$?
- This can be interpreted as the **training step**
- It is in general the most difficult problem

Summary: 4 Operations on HMMs

1. Compute data likelihood $p(Z|\theta)$ from a known model
    - Can be computed with the **forward** algorithm
2. Filtering or Smoothing of the state probability
    - Filtering: **forward** algorithm
    - Smoothing: **forward-backward** algorithm
3. Compute optimal state sequence with a known model
    - Can be computed with the **Viterbi**-Algorithm
4. Learn model parameters for an observation sequence
    - Can be computed using **Expectation-Maximization** (or Baum-Welch)

The Forward Algorithm

Goal: compute $p(Z|\theta)$ (we drop θ in the following)

$$p(\mathbf{z}_1, \dots, \mathbf{z}_n) = \sum_{\mathbf{x}_n} p(\mathbf{z}_1, \dots, \mathbf{z}_n, \mathbf{x}_n) =: \sum_{\mathbf{x}_n} \alpha(\mathbf{x}_n)$$

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We can calculate α recursively:

$$\alpha(\mathbf{x}_n) = p(\mathbf{z}_n \mid \mathbf{x}_n) \sum_{\mathbf{x}_{n-1}} \alpha(\mathbf{x}_{n-1}) p(\mathbf{x}_n \mid \mathbf{x}_{n-1})$$

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This is the same recursive formula as we had in the first lecture!

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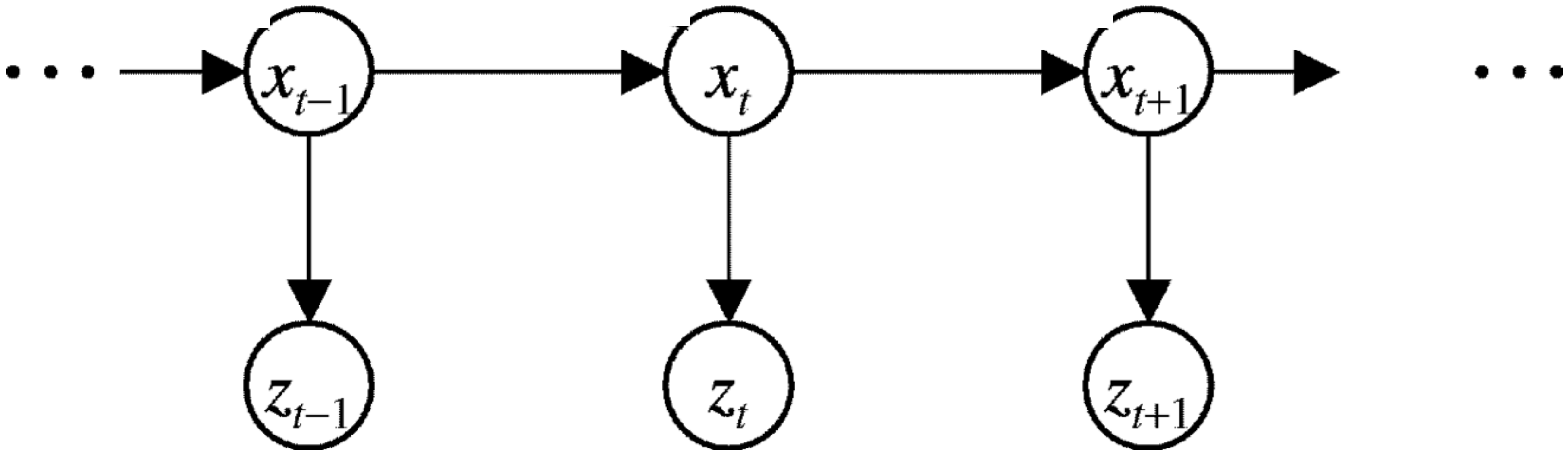
This is the same recursive formula as we had in the first lecture!

Filtering: $$p(\mathbf{x}_n \mid \mathbf{z}_1, \ldots, \mathbf{z}_n) = \frac{p(\mathbf{z}_1, \ldots, \mathbf{z}_n, \mathbf{x}_n)}{p(\mathbf{z}_1, \ldots, \mathbf{z}_n)} = \frac{\alpha(\mathbf{x}_n)}{\sum_{\mathbf{x}_n} \alpha(\mathbf{x}_n)}$$

The Forward-Backward Algorithm

- As before we set $\alpha(\mathbf{x}_t) = p(\mathbf{z}_1, \dots, \mathbf{z}_t, \mathbf{x}_t)$
- We also define $\beta(\mathbf{x}_t) = p(\mathbf{z}_{t+1}, \dots, \mathbf{z}_n \mid \mathbf{x}_t)$



The Forward-Backward Algorithm

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- We also define $\beta(\mathbf{x}_t) = p(\mathbf{z}_{t+1}, \ldots, \mathbf{z}_n \mid \mathbf{x}_t)$
- This can be recursively computed (backwards):

$$\beta(\mathbf{x}_t) = \sum_{\mathbf{x}_{t+1}} \beta(\mathbf{x}_{t+1}) p(\mathbf{z}_{t+1} \mid \mathbf{x}_{t+1}) p(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

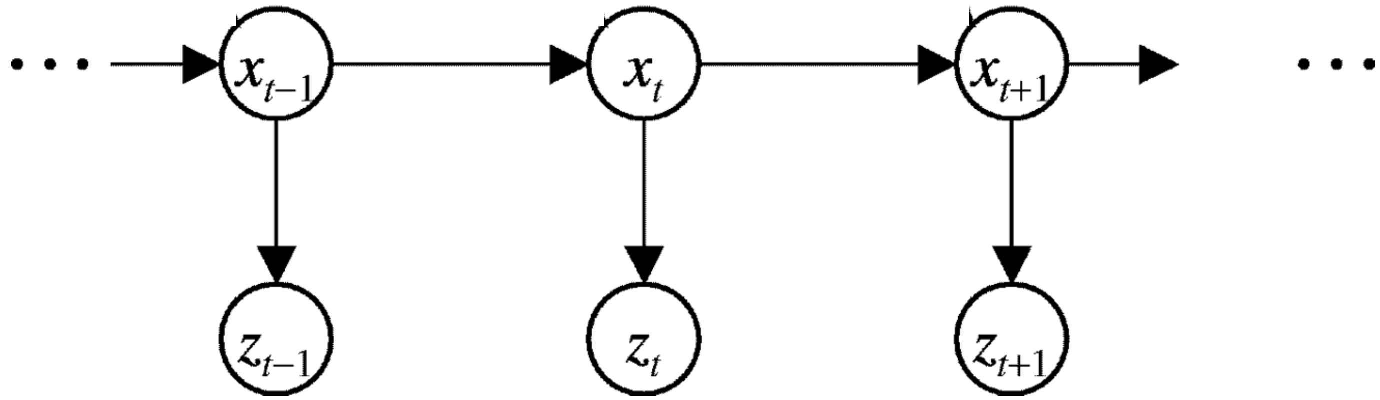
- This is exactly the same as the message-passing algorithm (sum-product)!
  - forward messages α_t (vector of length K)
  - backward messages β_t (vector of length K)

2. Computing the Most Likely States

- Goal: find a state sequence $x_1, x_2, x_3 \ldots$ that maximizes the probability $p(X, Z | \theta)$

- Define $\delta_t(j) := \max_{x_1, \ldots, x_{t-1}} p(\mathbf{x}_{1:t-1}, x_t = j \mid \mathbf{z}_{1:t})$

  This is the probability of state j by taking the most probable path.



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  This can be computed recursively:

$$\delta_t(j) := \max_i \delta_{t-1}(i) p(x_t \mid x_{t-1}) p(z_t \mid x_t)$$

  we also have to compute the argmax:

$$a_t(j) := \arg\max_i \delta_{t-1}(i) p(x_t \mid x_{t-1}) p(z_t \mid x_t)$$

The Viterbi algorithm

- Initialize:
  - $\delta(x_0) = p(x_0)\, p(z_0 \mid x_0)$
  - $\psi(x_0) = 0$
- Compute recursively for $n=1\ldots N$:
  - $\delta(x_n) = p(z_n|x_n) \max\limits_{x_{n-1}} [\delta(x_{n-1})\, p(x_n|x_{n-1})]$
  - $a(x_n) = \operatorname{argmax}\limits_{x_{n-1}} [\delta(x_{n-1})\, p(x_n|x_{n-1})]$
- On termination:
  - $p(Z,X|\theta) = \max\limits_{x_N} \delta(x_N)$
  - $x_N^* = \operatorname{argmax}\limits_{x_N} \delta(x_N)$
- Backtracking:
  - $x_n^* = a(x_{n+1}^*)$

3. Learning the Model Parameters

- Given an observation sequence $z_1, z_2, z_3 \ldots$
- Find optimal model parameters $\theta = \pi, A, \varphi$
- We need to maximize the likelihood $p(Z|\theta)$
- Can not be solved in closed form
- Iterative algorithm:
  Expectation Maximization (EM) or for the case of HMMs: Baum-Welch algorithm

The Baum-Welsh algorithm

- E-Step (assuming we know π,A,ϕ, i.e. θ^{old})
- Define the posterior probability of being in state i at step k:
- Define $\gamma(\mathbf{x}_n) = p(\mathbf{x}_n | Z)$

The Baum-Welsh algorithm

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- It follows that $\gamma(\mathbf{x}_n) = \alpha(\mathbf{x}_n)\, \beta(\mathbf{x}_n) \,/\, p(Z)$

The Baum-Welsh algorithm

- E-Step (assuming we know π,A,ϕ, i.e. θ^{old})
- Define the posterior probability of being in state i at step k:
- Define $\gamma(x_n) = p(x_n|Z)$
- It follows that $\gamma(x_n) = \alpha(x_n)\,\beta(x_n) / p(Z)$
- Define $\xi(x_{n-1}, x_n) = p(x_{n-1}, x_n|Z)$
- It follows that

  $\xi(x_{n-1}, x_n) = \alpha(x_{n-1}) p(z_n|x_n) p(x_n|x_{n-1}) \beta(x_n) / p(Z)$
- We need to compute:

  $Q(\theta, \theta^{old}) = \sum_X p(X|Z, \theta^{old}) \log p(Z, X|\theta)$

Expected complete data log-likelihood

The Baum-Welsh algorithm

- Maximizing Q also maximizes the likelihood:
  $p(Z|\theta) \geq p(Z|\theta^{old})$
- M-Step:

$$\pi_k = \frac{\sum_{\mathbf{x}} \gamma(\mathbf{x}) x_{1k}}{\sum_{j=1} \sum_{\mathbf{x}} \gamma(\mathbf{x}) x_{1j}}$$

here, we need forward and backward step!

$$A_{jk} = \frac{\sum_{t=2}^{T} \xi(x_{t-1,j}, x_{tk})}{\sum_{l=1}^{K} \sum_{t=2}^{T} \xi(x_{t-1,j}, x_{tl})}$$

- With these new values, Q is recomputed
- This is done until the likelihood does not increase anymore (convergence)

The Baum-Welsh algorithm - summary

- Start with an initial estimate of $\theta=(\pi,A,\phi)$
  e.g. uniformly and k-means for ϕ
- Compute $Q(\theta,\theta^{old})$ (E-Step)
- Maximize Q (M-step)
- Iterate E and M until convergence
- In each iteration one full application of the forward-backward algorithm is performed
- Result gives a **local** optimum
- For other local optima, the algorithm needs to be started again with new initialization

The Scaling problem

- Probability of sequences

$$\prod_i p(x_i \mid \ldots) << 1$$

<1

  - Probabilities are very small
  - The product of the terms soon is very small
- Usually: converting to log-space works
- But: we have sums of products!
- Solution: Rescale/Normalise the probability during the computation, e.g.:

$$\hat{\alpha}(x_n) = \alpha(x_n) \,/\, p(z_1, z_2, \ldots, z_n)$$

Summary

- HMMs are a way to model sequential data
- They assume discrete states
- Three possible operations can be performed with HMMs:
  - Data likelihood, given a model and an observation
  - Most likely state sequence, given a model and an observation
  - Optimal Model parameters, given an observation
- Appropriate scaling solves numerical problems
- HMMs are widely used, e.g. in speech recognition