Computer Vision Group
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Weekly Exercises 2

Room: 02.09.023
Wed, 04.05.2016, 14:00-16:00
Submission deadline: Tue, 03.05.2016, 23:59 to laehner@in.tum.de

Mathematics: Tangent spaces and Curvature

Exercise 1 (2 points). 1. Consider the following two coordinate maps of the sphere in $\mathbb{R}^3$.

$$\begin{aligned} x_1 : \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\} &\to \mathbb{R}^3 \\ (u,v) &\mapsto (u,v,\sqrt{1-u^2-v^2}) \qquad (1) \\ x_2 :]-10,10[\times]-10,10[&\to \mathbb{R}^3 \\ (u,v) &\mapsto \frac{1}{u^2+v^2+1}(2u,2v,u^2+v^2-1) \qquad (2) \end{aligned}$$

Calculate two vectors in $\mathbb{R}^3$ spanning the tangent space at $p = (2/3, 2/3, 1/3)$. Which vectors a, b in the domain of Dx_1, Dx_2 are mapped to the tangent vectors you gave before?

2. Proof that the tangent space $T_pM \subset \mathbb{R}^n$ of a submanifold $M \subset \mathbb{R}^n$ of dimension $m \leq n$ is a linear subspace and does not depend on the choice of the choosen coordinate map (x_i, U_i). (Lemma 3 from the Manifold chapter)

Solution. 1. First we calculate $x_1^{-1}(p) = (2/3, 2/3)$ and $x_2^{-1}(p) = (1,1)$ and

$$Dx_1(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{u}{\sqrt{1-u^2-v^2}} & -\frac{v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

$$Dx_2(u,v) = \begin{pmatrix} \frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2} & -\frac{4uv}{(u^2+v^2+1)^2} \\ -\frac{4uv}{(u^2+v^2+1)^2} & \frac{-2v^2+2u^2+2}{(u^2+v^2+1)^2} \\ \frac{4u}{(u^2+v^2+1)^2} & \frac{4v}{(u^2+v^2+1)^2} \end{pmatrix}$$

We get two vectors spanning the tangent space by multiplying two not linearly depend vectors with $Dx_1(x_1^{-1}(p))$ (or $Dx_2(x_2^{-1}(p))$).

$$\begin{aligned}
v_1 &= Dx_1(2/3, 2/3) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \\
v_2 &= Dx_1(2/3, 2/3) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}
\end{aligned}$$

Notice that v_1 and v_2 are not orthogonal like the original vectors but still span the tangent space. We already know the vectors that are mapped to v_1, v_2 by Dx_1 by construction and still have to find v_1' such that $v_1 = Dx_2(1,1)v_1'$. Solve for

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{4}{9} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

and similarly for v_2. It follows that $v_1 = Dx_2(1,1) \cdot \begin{pmatrix} -\frac{3}{2} \\ -3 \end{pmatrix}$ and $v_2 = Dx_2(1,1) \cdot \begin{pmatrix} -3 \\ -\frac{3}{2} \end{pmatrix}$.

2. That T_pM is a linear subspace follows directly from $Dx_i(u)$ being full-rank ($= m$). To show that the choice of the chart does not influence the tangent space we have to show that $\{Dx_i(u) \cdot v \mid \forall\ v \in \mathbb{R}^m\} = \{Dx_j(u) \cdot v \mid \forall\ v \in \mathbb{R}^m\}$.

$$\begin{aligned}
\tilde{v} \in \{Dx_i(u_i) \cdot v \mid \forall\ v \in \mathbb{R}^m\} &\Leftrightarrow \exists\ v \in \mathbb{R}^m : \tilde{v} = Dx_i(u_i) \cdot v \\
&\Leftrightarrow \exists\ v \in \mathbb{R}^m : \tilde{v} = \left(D(x_j \circ x_j^{-1} \circ x_i)\right)(u_i) \cdot v \\
&\Leftrightarrow \exists\ v \in \mathbb{R}^m : \tilde{v} = (Dx_j)\underbrace{(x_j^{-1} \circ x_i)(u_i)}_{=u_j} \cdot (Dx_j^{-1})\underbrace{(x_i(u_i))}_{=p} \cdot (Dx_i)(u_i) \cdot \\
&\Leftrightarrow \exists\ v' \in \mathbb{R}^m : \tilde{v} = (Dx_j)(u_j) \cdot v' \\
&\Leftrightarrow \tilde{v} \in \{Dx_j(u_j) \cdot v \mid \forall\ v \in \mathbb{R}^m\}
\end{aligned}$$

This holds if $v' = \underbrace{(Dx_j^{-1})(p) \cdot (Dx_i)(u_i)}_{2\times 2,\text{full-rank}} \cdot v$ is chosen.

Exercise 2 (1 point)**.** Find a coordinate map $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ of the torus

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 | \left(\sqrt{x^2+y^2} - a\right)^2 + z^2 = r^2 \right\}$$

with $a > r > 0$. (It can of course not cover the complete manifold.)

Solution. Since in the defining equation of T only the distance $\sqrt{x^2+y^2}$ in the xy-plane from the origin plays a role, the set T is symmetric under rotations around the z-axis. It is therefore sufficient to understand the set in the xz-plane - the hole torus then shows up via rotation around the z-axis. For $y=0$ we get

$$(|x|-a)^2 + z^2 = r^2.$$

Those are two circles with radius r with centers $(x=a, z=0)$ and $(x=-a, z=0)$. A parametrization can now be given via a rotation of one of them around the z-axis:

$$\mathbf{x} : (0, 2\pi)^2 \to \mathbb{R}^3, \quad (u,v) \mapsto \begin{pmatrix} \cos(v)(a + r\cos(u)) \\ \sin(v)(a + r\cos(u)) \\ r\sin(u) \end{pmatrix}$$

The coordinate map is smooth since $Dx(u)$ is of full rank:

$$Dx(u) = \begin{pmatrix} -r\cos(v)\sin(u) & -\sin(v)(a + r\cos(u)) \\ -r\sin(v)\sin(u) & \cos(v)(a + r\cos(u)) \\ r\cos(u) & 0 \end{pmatrix}$$

Exercise 3 (1 point)**.** The circle with radius r can be represented by the implicit formulation

$$\begin{aligned} \varphi :& \mathbb{R}^2 \to \mathbb{R} \\ x &\mapsto \sqrt{\langle x, x\rangle} \\ C_r &= \varphi^{-1}(r) \end{aligned}$$

Calculate the curvature on each point of C_r.

Solution.

$$\begin{aligned} \nabla\varphi &= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{pmatrix} \\ \frac{\nabla\varphi}{||\nabla\varphi||} &= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{pmatrix} \\ \operatorname{div}\left(\frac{\nabla\varphi}{||\nabla\varphi||}\right) &= \frac{x_1^2+x_2^2}{(x_1^2+x_2^2)^{3/2}} \\ &= \frac{r^2}{r^3} = \frac{1}{r} \end{aligned}$$

Exercise 4 (2 points)**.** Show that the push-forward is a linear mapping.

Solution. For a linear mapping we have to show that

$$\begin{aligned} Df(p)[v_1 + v_2] &= Df(p)[v_1] + Df(p)[v_2] \\ Df(p)[\lambda v] &= \lambda Df(p)[v] \end{aligned}$$

when $f : M \to N$.

We will use the definition of the push-forward using curves but the proof can as easily (maybe even easierly) with the differentials of x and f. Let $x : U \to M$ be a coordinate map w.l.o.g. $0 \in U \subset \mathbb{R}^d$ and $x(0) = p$ as well as $f(p) = q$. Let $u, v \in T_pM$ then there exist $h_u, h_v \in \mathbb{R}^d$ such that

$$\begin{aligned} c_u :\ &(-\epsilon, \epsilon) \to M & t &\mapsto x(t \cdot h_u) \\ c_v :\ &(-\epsilon, \epsilon) \to M & t &\mapsto x(t \cdot h_v) \end{aligned}$$

They define the equivalence classes $[c_u] = u$ and $[c_v] = v$ for which hold that $c_{u/v}(0) = p$ and $Dc_{u/v}(0) = u/v$. Further, we have

$$\begin{aligned} f \circ c_u :\ &(-\epsilon, \epsilon) \to N \\ f \circ c_v :\ &(-\epsilon, \epsilon) \to N \end{aligned}$$

and it follows by definition that $[f \circ c_u] \in T_qN$ and $[f \circ c_v] \in T_qN$. We define $[c_u] + [c_v] = [c_{u+v}]$ by

$$\begin{aligned} c_{u+v} : &(-\epsilon, \epsilon) \to M \\ &t \mapsto x(t \cdot (h_u + h_v)) \\ c_\lambda : &(-\epsilon, \epsilon) \to M \\ &t \mapsto x(t \cdot \lambda h_u) \end{aligned}$$

$$\begin{aligned} Df(p)[c_{u+v}] &= [f \circ c_{u+v}] \\ &= [(f \circ x)(t \cdot (h_u + h_v))] \\ &= \frac{\partial}{\partial t}((f \circ x)(t \cdot (h_u + h_v))) \mid_{t=0} \\ &= \frac{\partial (f \circ x)}{\partial t}(t \cdot (h_u + h_v)) \cdot (h_u + h_v) \mid_{t=0} \\ &= \frac{\partial (f \circ x)}{\partial t}(0) \cdot h_u + \frac{\partial (f \circ x)}{\partial t}(0) \cdot h_v \\ &= [f \circ c_u] + [f \circ c_v] \\ &= Df(p)[c_u] + Df(p)[c_v] \end{aligned}$$

and

$$
\begin{aligned}
Df(p)[c_\lambda] &= [f \circ c_\lambda] \\
&= [(f \circ x)(t \cdot \lambda h_u)] \\
&= \frac{\partial}{\partial t}(f \circ x)(t \cdot \lambda h_u) \mid_{t=0} \\
&= \frac{\partial (f \circ x)}{\partial t}(t \cdot \lambda h_u) \cdot \lambda h_u \mid_{t=0} \\
&= \lambda \cdot \frac{\partial (f \circ x)}{\partial t}(0) h_u \\
&= \lambda [f \circ c_u] \\
&= \lambda \cdot Df(p)[c_u]
\end{aligned}
$$

Programming: Curvature

Download the supplementary material from the homepage. It contains some black-white silhouette images from the MPEG7 dataset (http://www.dabi.temple.edu/˜shape/MPEG7/dataset.html) and a file to extract contour information from these images.

Exercise 5 (2 points)**.** Read out the image files `bat-9.gif`, `device7-1.gif`, `turtle-1.gif` into matrices (use `imread`, it reads positive integers. Changing the type to double will help). Include an image for each sub-exercise in your solution sheet.

1. Calculate the curvature on the contour. It can be seen as the level set function somewhere between 0 and 1 so the formula $\kappa(p) = \operatorname{div}\left(\frac{\nabla F(\cdot)}{||\nabla F(\cdot)||}\right)(p)$ from the lecture can be used. There are `imgradient` and `imgradientxy` in Matlab or implement your own finite difference gradient as an exercise (it's quite easy). The divergence can be calculated with `divergence`.

2. The result from the last exercise was pretty ugly. The reason is that the function we considered was not smooth but went zig-zag along the edges of the pixels. Use a gaussian filter on the image before calculating the curvature. (See `imgaussfilt`) Use the `extract_contour.m` from the supplementary material to get a binary mask for the contour with different thickness. Play around with different σ for the filter and thicknesses of the mask.

Exercise 6 (1 point)**.** In most applications we want to find out which shapes are similar to each other. Create a descriptor for each shape in the supplementary material and create a histogram of the different curvatures on the contour (don't forget to normalize because normally the contours will not have the same amount of points). There is a Matlab function `histogram` if you are not familiar with histograms.
Compare the descriptor of `device7-1` to all other descriptors (for example with the Euclidean distance) and sort the remaining shapes in order of similarity to device7-1.