Analysis of Three-Dimensional Shapes
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Weekly Exercises 4

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Mathematics: Linear Algebra Recap

Let X be a vector space. An *inner product* is a function $f : X \times X \to \mathbb{C}$ with the following properties:

1. $f(x,x) \geq 0 \quad \forall\, x \in X$ and $f(x,x) = 0 \Leftrightarrow x = 0$
2. $f(x,y) = \overline{f(y,x)}$
3. $f(x + \alpha x', y) = f(x,y) + \alpha f(x', y) \quad \forall\, x, x', y \in X, \alpha \in \mathbb{C}$

The standard inner product is defined as $\langle x, y\rangle = x^\top \bar{y},\ x, y \in \mathbb{R}^n$. If $M \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix an M-inner product can be defined by taking $\langle x, y\rangle_M = x^\top M y$.

A linear operator $T : X \to X$ is called *self-adjoint* w.r.t. an inner product if the following holds:

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

An *eigenvector* is an element $0 \neq x \in X$ for which there exists a scalar $\lambda \in \mathbb{C}$ such that

$$Tx = \lambda x$$

The scalar λ is called *eigenvalue*.

Exercise 1. Let $L = M^{-1}S \in \mathbb{R}^{n\times n}$ be self-adjoint w.r.t. the M inner product. Show that the following statements hold.

1. S is symmetric (self-adjoint) w.r.t. to the standard inner product.
2. The eigenvalues of L are real.
3. The eigenvectors v_i, v_j with respective eigenvalues $\lambda_i \neq \lambda_j$ are orthogonal.
4. $v_1, ..., v_k$ are eigenvectors of L with the same eigenvalue λ, then $\sum_i \alpha_i v_i$ is also an eigenvector with eigenvalue λ.

Solution. 1.

$$\begin{aligned}\langle Sx, y\rangle &= x^\top S^\top \bar{y} = x^\top S^\top M^{-1} M y \\ &= \langle M^{-1} Sx, y\rangle_M = \langle Lx, y\rangle_M \\ &= \langle x, Ly\rangle_M = x^\top M M^{-1} S \bar{y} \\ &= \langle x, Sy\rangle\end{aligned}$$

2. Let λ be an eigenvalue of L such that $Lv = \lambda v$.

$$\begin{aligned}\lambda\langle v, v\rangle_M &= \langle \lambda v, v\rangle_M \\ &= \langle Lv, v\rangle_M = \langle v, Lv\rangle \\ &= \langle v, \lambda v\rangle \\ &= \bar{\lambda}\langle v, v\rangle\end{aligned}$$

3.

$$\begin{aligned}\lambda_i\langle v_i, v_j\rangle &= \langle \lambda_i v_i, v_j\rangle \\ &= \langle Lv_i, v_j\rangle = \langle v_i, Lv_j\rangle \\ &= \langle v_i, \lambda_j v_j\rangle \\ &= \lambda_j\langle v_i, v_j\rangle\end{aligned}$$

Since we assumed $\lambda_i \neq \lambda_j$ this is only possible when $\langle v_i, v_j\rangle = 0$.

4.

$$\begin{aligned}L\left(\sum_i \alpha_i v_i\right) &= \sum_i \alpha_i L v_i \\ &= \sum_i \alpha_i \lambda v_i \\ &= \lambda\left(\sum_i \alpha_i v_i\right)\end{aligned}$$