Analysis of Three-Dimensional Shapes
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Weekly Exercises 7

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Mathematics: Euler Characteristic

Exercise 1 (2 points)**.** Given a coordinate map $x\colon U \to M$ of a surface M with respect to a convex coordinate domain U. Assume further that given $p_1 = x(u_1) \in M$ and $p_2 = x(u_2) \in M$, the curves $\gamma_1, \gamma_2\colon [0,1] \to U$ that connect u_1 and u_2 are parametrizations of the geodesics $x \circ \gamma_i$ between p_1 and p_2. Show that non-positive Gaussian curvature $K(p) < 0\ \forall p \in M$ implies $\gamma_1 = \gamma_2$.

Solution. Let $q = (x \circ \gamma_1)(i), i \in]0,1[$ and $T = (p_1, p_2, q)$ a triangle on M connected by the edges described by $(x \circ \gamma_1), (x \circ \gamma_2)$. Notice that the geodesic curvature on both is zero and the inner angle at point q is π. Furthermore, let β_1, β_2 be the inner angles between the curves at p_1, p_2.

We use the Gauss-Bonnet Theorem:

$$\begin{aligned} 2\pi &= \int_T K(p)\mathrm{d}p + \underbrace{\int_{\delta T} \kappa_g(p)\mathrm{d}p}_{=0} + \sum_{i<3} \alpha_i \\ &= \int_T K(p)\mathrm{d}p + 3\pi - \sum_{i<3} \alpha_i \\ &= \int_T K(p)\mathrm{d}p + 3\pi - \beta_1 - \beta_2 - \pi \\ \Rightarrow \underbrace{\beta_1 + \beta_2}_{\geq 0} &= \underbrace{\int_T K(p)\mathrm{d}p}_{\leq 0} \end{aligned}$$

This can only hold if $\beta_1 + \beta_2 = 0$ which implies that $\gamma_1 = \gamma_2$ and the area of T is zero which implies also the integral is zero.

Exercise 2 (2 points)**.**

1. Given the triangulation (V, E, F) of a closed surface of Euler characteristic χ. Express $|V|$ and $|E|$ with respect to χ and $N := |F|$.
2. Show that the Euler characteristic of a surface is $|V| - |E| + |F|$ even if the faces of the mesh (V, E, F) are not necessarily triangles, but convex polygons.

Solution.

1. Each face has 3 edges, each edge appears twice:

$$e = \frac{3}{2} f$$

Plug both into the Euler formula:

$$v = \chi + \frac{1}{2}f$$

2. We can represent convex n-sided polygon as n triangles by putting an additional vertex in the middle. This which increases the number of faces by $n-1$ and the number of edges by n and the number of vertices 1 for each polygon. k represents any polygon with k sides.

$$\chi = (v + \sum_{k>3} 1) - (e + \sum_{k>3} k) + (f + \sum_{k>3}(k-1)) = v - e + f$$

This proof is also nice with induction.

Programming: Geodesics

Download the supplementary material 07 from the homepage. It contains a MinHeap class file, `incidenceMatrix.m`, `sheet7check.m`, `dijkstra.m` and `fastmarching.m` with the basic constructions of the algorithms (and some other things to let you check your solutions). If you do not know how a Heap works, I suggest you make yourself familiar with it. Cheat sheet for the functionality:

1. `MinHeap = MinHeap(int)` creates a new instance of the Heap with maximum size n (and maximum key n)
2. `bool = MinHeap.decrease(key, value)` decreases the value of the given key, nothing happens if the key does not exist in the heap or the value is larger than the previous value, the return value indicates if the heap was changed during the operation
3. `bool = MinHeap.increase(key, value)` like decrease basically
4. `double = MinHeap.peakKey(key)` returns the value of the given key, -1 if the key is invalid
5. `[value, key] = MinHeap.pop()` returns key and minimum value in the heap. $(-1, -1)$ if the heap is empty
6. `MinHeap.push(value, key)` adds a new (key, value) tupel to the heap. Nothing happens if the key already exists in the heap
7. `bool = MinHeap.isEmpty()` returns true if the heap is empty

You can check if your calculations are correct by running the scripts in `sheet7check.m`.

Exercise 3 (3 points)**.** Implement the geodesic distance and path between two points a, b on a manifold with the dijkstra algorithm introduced in the lecture. The file `dijkstra.m` already contains the cornerstones such that you only have fill in the

dijkstra step and extract the solution. Additionally to the distance, we also want to get the shortest path. For that we simply need to keep track of the predecessor of every vertex that resulted in an update of the minimal distance there. In the end, we need to backtrack through the predecessors from b until we reach a.

Exercise 4 (5 points)**.** Implement the distance map of one point of a manifold using the fast marching algorithm from the lecture. `march.m` is meant to include the update step which is the only real difference to dijkstra.An assumption in the lecture was that the triangle is aligned to the x-y-plane, the code in `march.m` already includes this alignment. If you are getting confused, use the B, R, G sets as in the lecture but this is in general not necessary and slows down the code. You do not have to implement the pre-processing step eliminating obtuse triangles.