Computer Vision Group
Institut für Informatik
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Weekly Exercises 8

Room: 02.09.023

Wed, 06.07.2016, 14:00-16:00

Submission deadline: Tue, 05.07.2016, 23:59 to laehner@in.tum.de

Mathematics: Functional Maps

Exercise 1 (2 points)**.** Let M, N be two 2-dimensional manifolds and ϕ_i^M, ϕ_j^N their respective eigenfunctions of the LBO. Let $T : M \to N$ be a mapping between M and N and T_F its corresponding Functional Map denoted by C in matrix form for the bases Φ^M, Φ^N. Show the following:

1. If T is area preserving, it holds that $C^\top C = Id$.
2. If T is conformal, it holds that $\Lambda_M = C^\top \Lambda_N C$ where $\Lambda_{M,N}$ are diagonal matrices with the eigenvalues of the LBO as diagonal entries.

Solution. Let $f = \Phi_M \alpha$ and $g = \Phi_M \beta$.

1. If T is area preserving it holds that $\int_M f(x)g(x)\,dx = \int_N T_F(f)(y)T_F(g)(y)\,dy$.

$$\begin{aligned}\int_M f(x)g(x)\ \mathrm{d}x &= (\Phi_M\alpha)^\top M_M \Phi_M \beta \\ &= \alpha^\top \Phi_M^\top M_M \Phi_M \beta = \alpha^\top \beta\end{aligned}$$

$$\begin{aligned}\int_N T_F(f)(y)T_F(g)(y)\ \mathrm{d}y &= (\Phi_N C\alpha)^\top M_N \Phi_N C\beta \\ &= \alpha^\top C^\top \Phi_N^\top M_N \Phi_N C \beta = \alpha^\top C^\top C\beta\end{aligned}$$

   For both equations to be equal $C^\top C$ has to be the identity.

2. If T is area preserving it holds that $\int_M \langle \nabla f(x), \nabla g(x)\rangle\,dx = \int_N \langle \nabla T_F(f)(y), \nabla T_F(g)(y)\rangle\,dy$.

$$\begin{aligned}\int_M \langle \nabla f(x), \nabla g(x)\rangle\ dx &= -\int_M \langle f(x), \Delta g(x)\rangle\ dx \\ &= (\Phi_M\alpha)^\top M_M \underbrace{\Phi_M \Lambda_M \Phi_M^{-1}}_{=L} \Phi_M\beta \\ &= \alpha^\top \Lambda_M \beta\end{aligned}$$

$$\begin{aligned}\int_N \langle \nabla T_F(f)(y), \nabla T_F(g)(y)\rangle\ dy &= -\int_N \langle T_F(f)(y), \Delta T_F(g)(y)\rangle\ dy \\ &= (\Phi_N C\alpha)^\top M_N \Phi_N \Lambda_N \Phi_N^{-1} \Phi_N C\beta \\ &= \alpha^\top C^\top \Lambda_N C\beta\end{aligned}$$

   Imposing equality on both results in $\Lambda_M = C^\top \Lambda_N C$.

Programming: Geodesics and Heat Diffusion

Exercise 2 (3 points)**.** In the previous exercises you implemented the mass matrix M and the stiffness matrix C which can be composed to the Laplace-Beltrami Operator $L = CM^{-1}$.

1. Calculate the eigenvalues and functions of the LBO by solving the generalized eigenvalue problem $C\phi = \lambda M\phi$. This can be done with the Matlab `eigs` function. You have to specify that you want to solve for the smallest eigenvalues (the largest is default). Visualize the first 20 eigenfunctions and find out how their frequency changes.

2. Implement heat diffusion. Start with one unit of heat at one vertex and zero everywhere else and observe how the heat distributes over time. Also try out different numbers of eigenfunctions (20, 50, 100, 200). Remember that the LBO eigenfunctions are only orthogonal w.r.t. the M-inner product.

3. Implement the Heat Kernel Signature. Compare it on different shapes (with different poses) and different time values on the TOSCA data set. You can find it here: http://tosca.cs.technion.ac.il/book/resources_data.html (choose TOSCA high-resolution)