

Analysis of 3D Shapes (IN2238)

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2. Manifolds and Shapes

Dimension

Dimension of Linear Spaces

Conceptually, we have a very good understanding what a **dimension** is. Sometimes, we refer to it as **degrees of freedom**.

Even for (linear) \mathbb{R} -vector spaces, the definition can be quite involved:

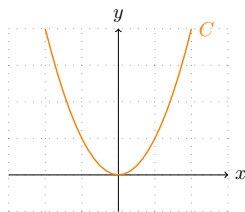
$$\dim(U) = \min_{\substack{B \subset U \\ \langle B \rangle = U}} |B|$$

$$\langle B \rangle = \left\{ x \mid x = \sum_{b \in B} \lambda_b \cdot b, \lambda \in \mathbb{R}^B \right\}$$

For linear vector spaces U , it suffices to find a linear bijection $\Phi: \mathbb{R}^d \rightarrow U$ in order to prove that U is a vector space of dimension d .

Here, U can be an **arbitrary \mathbb{R} -vector space**. It does not need to be **represented as a subset of an \mathbb{R}^N** .

Dimension of Curves



Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$$

is not a linear space, we like to think of it as a one-dimensional object. Why?

Dimension of Curves

Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$$

is not a linear space, we can find a bijection

$$\varphi: \mathbb{R} \rightarrow C \quad t \mapsto (t, t^2)$$

that introduces a one-dimensional coordinate $t \in \mathbb{R}$ to each $(x, y) \in C$.

For what kind of sets C can we define a "curved" dimension?

What kind of functions φ guarantee a unique dimension?

Continuous Mappings

Definition 1 (Open Set). A set $O \subset \mathbb{R}^n$ is called open iff

$$x \in O \quad \Rightarrow \quad \exists \varepsilon > 0 : B_\varepsilon(x) \subset O,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}$ is a ball of radius ε centered at $x \in \mathbb{R}^n$.

Definition 2 (Relatively Open Set). Given a subset $X \subset \mathbb{R}^n$, we call $O \subset X$ relatively open iff there exists an open set $\hat{O} \subset \mathbb{R}^n$ of \mathbb{R}^n such that

$$O = \hat{O} \cap X$$

The set $\mathcal{T}(X)$ of all relatively open subsets is called the topology of X .

Definition 3 (Continuous Mappings). Given subsets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, we call a mapping $f: X \rightarrow Y$ continuous iff

$$O \in \mathcal{T}(Y) \quad \Rightarrow \quad f^{-1}(O) \in \mathcal{T}(X)$$

Homeomorphism

Theorem 1 (Cantor, 1877). Given the interval $I = [0, 1]$, there is a bijection $\varphi: I \rightarrow I^2$.

Theorem 2 (Peano, 1890). Given the interval $I = [0, 1]$, there is a continuous bijection $\varphi: I \rightarrow I^2$.

Definition 4. A bijection $\varphi: X \rightarrow Y$ is called a **homeomorphism** iff φ and φ^{-1} are continuous.

Theorem 3 (Brouwer, 1911). The existence of a homeomorphism $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ implies $m = n$.

In practice, it is often difficult to check whether a function is continuous. To check differentiability is on the other hand easier. Thus, we would like to extend the idea of homeomorphisms to the class of differentiable functions.

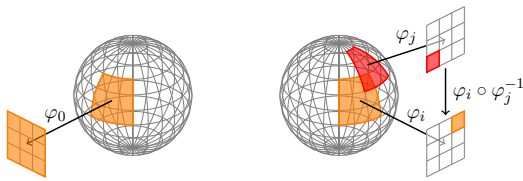
Definition 5. A bijection $\varphi: X \rightarrow Y$ is called a **C^k -diffeomorphism** iff φ and φ^{-1} are C^k -functions, i.e., for all $i \leq k$ exist the i -th derivatives of these functions and they are continuous. C^∞ -diffeomorphisms are called **(smooth) diffeomorphisms**.

Problem: While we know how to "extend" the idea of dimensions to \mathbb{R}^n **non-linearly**, we still need to define when $C \subset \mathbb{R}^2$ is a 1D object.

Solution: Concept of **manifolds** and **sub-manifolds**.

Manifold

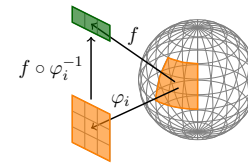
Chart and Atlas



Definition 6. (M_0, φ_0) is called an **n -dimensional chart** of M iff $\varphi_0: M_0 \rightarrow U_0$ is a homeomorphism between the open sets $M_0 \subset M$ and $U_0 \subset \mathbb{R}^n$.

Definition 7. A collection $(M_i, \varphi_i)_{i \in \mathcal{I}}$ of charts is called a **C^k atlas** iff $M = \bigcup_{i \in \mathcal{I}} M_i$ and for any two charts φ_i and φ_j , the mapping $\varphi_i \circ \varphi_j^{-1}$ is a C^k -diffeomorphism between $\varphi_j(M_i \cap M_j)$ and $\varphi_i(M_i \cap M_j)$.

Smooth Functions



Definition 8. A set M with a C^∞ atlas $(M_i, \varphi_i)_{i \in \mathcal{I}}$ is called a **(smooth) manifold**.

Definition 9. Given a manifold M , a function $f: M \rightarrow \mathbb{R}^d$ is called **smooth** iff for all charts (M_i, φ_i) , the function $f \circ \varphi_i^{-1}: \varphi_i(M_i) \rightarrow \mathbb{R}^d$ is smooth.

Whether a function is smooth depends very much on the chosen atlas. Therefore, we are more interested in submanifolds.

Generalization of \mathbb{R}^n

We like to think of n -dimensional manifolds as extensions of the linear space \mathbb{R}^n . In order to see this we will define the "manifold" \mathbb{R}^n .

Let $(M_i)_{i \in \mathcal{I}}$ be a collection of open sets $M_i \subset \mathbb{R}^n$ that cover \mathbb{R}^n . Choices are:

$$\begin{aligned} \mathcal{I} &= \mathbb{Z}^n & M_{(i_1, \dots, i_n)} &= \{x \in \mathbb{R}^n \mid \|x - i\| < \sqrt{n}\} \\ \mathcal{I} &= \{0\} & M_0 &= \mathbb{R}^n \end{aligned}$$

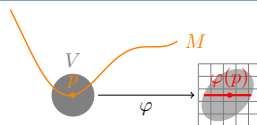
Given these sets, the charts can be chosen as (M_i, id_{M_i}) and for overlapping charts, we obtain the diffeomorphism

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}: M_i \cap M_j &\rightarrow M_i \cap M_j \\ p &\mapsto p \end{aligned}$$

With these atlases, we obtain that the "smooth functions" f on the "manifold" \mathbb{R}^n are exactly the functions $C^\infty(\mathbb{R}^n)$.

Submanifold

Smooth Submanifold



Definition 10. A subset $M \subset \mathbb{R}^n$ is a **(smooth) submanifold** of dimension m iff for every point $p \in M$, there exists a chart (V, φ) of \mathbb{R}^n such that

- $p \in V$.
- $\varphi(M \cap V) = \mathbb{R}^m \cap \varphi(V)$.

We call $n - m$ the **co-dimension** of M .

Note that φ is automatically a diffeomorphism. Why?
Note that M is automatically a manifold. Why?

Implicit Function Theorem

Theorem 4 (Implicit Function Theorem). Given a C^k -mapping $F: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $(x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $F(x_0, y_0) = 0$ and $\frac{\partial}{\partial y} F(x_0, y_0)$ is invertible. Then there exists $U \in \mathcal{T}(\mathbb{R}^m)$, $V \in \mathcal{T}(\mathbb{R}^n)$ and $\varphi: U \rightarrow V$ such that

- $x_0 \in U$, $y_0 \in V$ and $\varphi(x_0) = y_0$.
- For all $x \in U$ we have $F(x, \varphi(x)) = 0$.
- φ is a bijective C^k -mapping.
- For all $x \in U$ we have $\frac{\partial}{\partial x} \varphi(x) = - \left[\frac{\partial}{\partial y} F(x, \varphi(x)) \right]^{-1} \frac{\partial}{\partial x} F(x, \varphi(x))$.

Theorem 5 (Inverse Function Theorem). Given a C^k -mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Df(x_0)$ is of maximal rank for a $x_0 \in \mathbb{R}^n$. Then there exists $U \in \mathcal{T}(\mathbb{R}^n)$, $V \in \mathcal{T}(\mathbb{R}^n)$ such that $f: U \rightarrow V$ is a C^k -diffeomorphism and $D(f^{-1})(f(x)) = (Df)^{-1}(x)$

Proof. Define $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $F(x, y) = x - f(y)$ and apply the *Implicit Function Theorem*. \square

Explicit Submanifolds

Dimension Manifold Submanifold Tangent Space

One important example of a submanifold is described by smooth coordinate mappings $x: U \rightarrow \mathbb{R}^n$:

Lemma 1. A subset $M \subset \mathbb{R}^n$ together with smooth coordinate mappings $(x_i, U_i)_{i \in \mathcal{I}}$ is a smooth submanifold of dimension m if the following holds:

- All U_i are open subsets of \mathbb{R}^m .
- $M = \bigcup_{i \in \mathcal{I}} x_i(U_i)$.
- For all $u \in U_i$, x_i is smooth and $Dx_i(u) \in \mathbb{R}^{n \times m}$ is of maximal rank m .

Proof. Given $p \in M$, we choose $i \in \mathcal{I}$ and $\hat{u} \in \mathbb{R}^m$ such that $p = x_i(\hat{u})$. Since $Dx_i(\hat{u})$ is of maximal rank, we can find a matrix $A_0 \in \mathbb{R}^{n \times (n-m)}$ such that $A := (Dx_i(\hat{u}) \quad A_0) \in \mathbb{R}^{n \times n}$ is of maximal rank n .

Explicit Submanifolds

Dimension Manifold Submanifold Tangent Space

Proof (Cont.) As a result, we can define the smooth function

$$\psi: U_i \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n \\ (u_1, \dots, u_m, v_1, \dots, v_{n-m}) \mapsto \varphi(u) + A \cdot v$$

with $\det(D\psi(\hat{u}, 0)) \neq 0$. Using the *Inverse Function Theorem* proves the lemma. □

Now we can prove that

$$C := \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$$

is a manifold of dimension 1. How?

Implicit Submanifolds

Dimension Manifold Submanifold Tangent Space

In practice, it is often difficult to define different charts or coordinate functions. Instead, we like to define the manifold M by formulating certain constraints, e.g.,

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid \langle x, x \rangle = 1\}$$

Lemma 2. Given a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ and a regular value $c \in \mathbb{R}^k$, i.e.,

$$x \in f^{-1}(c) \quad \Rightarrow \quad \text{rank}(Df(x)) = k.$$

Then, $M := f^{-1}(c)$ is a submanifold of co-dimension k .

Implicit Submanifolds

Dimension Manifold Submanifold Tangent Space

Proof. Let $p \in M \subset \mathbb{R}^n$. Since $Df(p)$ is of rank k , we can find k columns of $Df(p)$ that are linear independent. W.l.o.g. we assume that these k columns are the last k . Thus, the function $f: \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ satisfies the *Implicit Function Theorem* with respect to $(x_0, y_0) = p$.

The implicit function $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ defines a coordinate mapping $x: \mathbb{R}^k \rightarrow \mathbb{R}^n$ via $x(u) := (u, \varphi(u))$, which proves the lemma.

Note that the implicit submanifold can be transformed into an explicit submanifold.

Given a point $p \in M$ the created coordinate mapping x satisfies

$$Dx(p) = \begin{pmatrix} \text{id} & \\ - \left[\frac{\partial f}{\partial y}(p) \right]^{-1} \frac{\partial f}{\partial x}(p) & \end{pmatrix}$$

Objects and Shapes

Dimension Manifold Submanifold Tangent Space

With these definitions in place, we can finally define what we mean by an object and a shape

Definition 11 (Object). An **object** of dimension d is an open subset $X \subset \mathbb{R}^d$ such that its boundary $B := \partial X$ is a submanifold of dimension $d - 1$. We will use the notation \mathcal{O}^d for the space of all objects of dimension d .

Some authors only study the boundary of an object.

Definition 12 (Shape). Given an equivalence relation \sim of \mathcal{O}^d , the equivalence class $[O]$ of an object $O \in \mathcal{O}^d$ is its shape. We call the set of all shapes the **shape space** \mathcal{O}^d / \sim .

For now we will focus on an equivalence relation that uses a combination of rotation, translation and scaling to define equivalent objects.

Tangent Space

Dimension Manifold Submanifold Tangent Space



Tangent Space

Dimension Manifold Submanifold Tangent Space

Since we want to focus on smooth submanifolds M of dimension m , we like to approximate the direct vicinity of a point $p \in M$ with a linear vector space of dimension m . This leads to the concept of the tangent space.

Definition 13 (Tangent Space). Let $M \subset \mathbb{R}^n$ be a submanifold of dimension $m \leq n$ that is given via coordinate functions $(x_i, U_i)_{i \in \mathcal{I}}$. Given $i \in \mathcal{I}$ such that $p = x_i(u)$, we define the **tangent space** $T_p M$ of M at the position p as

$$T_p M := \{Dx_i(u) \cdot v \mid v \in \mathbb{R}^m\} \quad [= \text{Im}(Dx_i(u))]$$

Lemma 3 (Tangent Space). The tangent space $T_p M \subset \mathbb{R}^n$ is a linear subspace and does not depend on the choice of the chosen coordinate map (x_i, U_i) .

Proof. Exercise. □



Tangent Space

Dimension Manifold Submanifold Tangent Space

Lemma 4 (Tangent Space). Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a smooth function with regular value $c \in \mathbb{R}^k$ and $M := f^{-1}(c)$ the manifold with respect to this value. For every $p \in M$ we have

$$T_p M := \{v \in \mathbb{R}^n \mid Df(p) \cdot v = 0\} \quad [= \ker(Df(p))]$$

Proof. The coordinate mapping x that we constructed for a implicitly defined manifold is $Dx(u) = \begin{pmatrix} \text{id} & \\ - \left[\frac{\partial f}{\partial y}(p) \right]^{-1} \frac{\partial f}{\partial x}(p) & \end{pmatrix}$. Using linear algebra, we obtain

$$y \in T_p M^\perp = \text{Im}(Dx(u))^\perp = \ker(Dx(u)^\top) \\ \Rightarrow 0 = y_1 - \left[\frac{\partial f}{\partial x}(p) \right]^\top \left[\frac{\partial f}{\partial y}(p) \right]^{-\top} y_2$$

Choosing $y_2 = \left[\frac{\partial f}{\partial y}(p) \right]^\top \cdot v$ leads to $y_1 = \left[\frac{\partial f}{\partial x}(p) \right]^\top \cdot v$, which proves $T_p M = \text{Im}(Df(p)^\top)^\perp = \ker(Df(p))$.



Dimension

- Peano, *Sur une courbe, qui remplit toute une aire plane*, 1890, Math. Annalen (36), 157–160.
- Brouwer, *Beweis der Invarianz der Dimensionenzahl*, 1911, Math. Annalen (70), 161–165.

Smooth Manifolds

- Poincaré, *Analysis Situs*, 1895, Journal de l'École Polytechnique.