Analysis of 3D Shapes (IN2238)

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Dimension of Linear Spaces

Conceptually, we have a very good understanding what a **dimension** is.
Sometimes, we refer to it as **degrees of freedom**.

Even for (linear) $\mathbb{R}$-vector spaces, the definition can be quite involved:

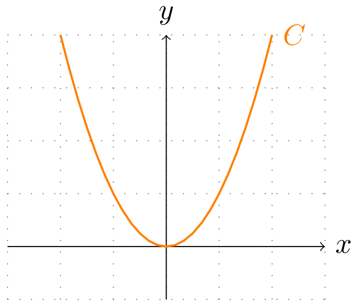
$$\dim(U) = \min_{\substack{B \subset U \\ \langle B \rangle = U}} |B|$$

$$\langle B \rangle = \left\{ x \,\middle|\, x = \sum_{b \in B} \lambda_b \cdot b, \lambda \in \mathbb{R}^B \right\}$$

For linear vector spaces U, it suffices to find a linear bijection $\Phi : \mathbb{R}^d \to U$ in order to prove that U is a vector space of dimension d.

Here, U can be **an arbitrary $\mathbb{R}$-vector space**. It does not need to be **represented as a subset of an $\mathbb{R}^N$**.

Dimension of Curves



Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 | y = x^2\}$$

is not a linear space, we like to think of it as a one-dimensional object. Why?

Dimension of Curves

Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 | y = x^2\}$$

is not a linear space, we can find a bijection

$$\varphi \colon \mathbb{R} \to C \qquad t \mapsto (t, t^2)$$

that introduces a one-dimensional coordinate $t \in \mathbb{R}$ to each $(x, y) \in C$.

For what kind of sets C can we define a "curved" dimension?

What kind of functions φ guarantee a unique dimension?

Continuous Mappings

Definition 1 (Open Set)**.** A set $O \subset \mathbb{R}^n$ is called open iff

$$x \in O \qquad \Rightarrow \qquad \exists \varepsilon > 0 : B_\varepsilon(x) \subset O,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n | \, \|x - y\| < \varepsilon\}$ is a ball of radius ε centered at $x \in \mathbb{R}^n$.

Definition 2 (Relatively Open Set)**.** Given a subset $X \subset \mathbb{R}^n$, we call $O \subset X$ relatively open iff there exists an open set $\hat{O} \subset \mathbb{R}^n$ of $\mathbb{R}^n$ such that

$$O = \hat{O} \cap X$$

The set $\mathcal{T}(X)$ of all relatively open subsets is called the topology of X.

Definition 3 (Continuous Mappings)**.** Given subsets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, we call a mapping $f \colon X \to Y$ continuous iff

$$O \in \mathcal{T}(Y) \qquad \Rightarrow \qquad f^{-1}(O) \in \mathcal{T}(X)$$

Homeomorphism

Theorem 1 (Cantor, 1877)**.** *Given the interval $I = [0, 1]$, there is a bijection $\varphi : I \to I^2$.*

Theorem 2 (Peano, 1890)**.** *Given the interval $I = [0, 1]$, there is a continuous bijection $\varphi : I \to I^2$.*

Definition 4. A bijection $\varphi\colon X \to Y$ is called a **homeomorphism** iff φ and φ^{-1} are continuous.

Theorem 3 (Brouwer, 1911)**.** *The existence of a homeomorphism $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ implies $m = n$.*

Diffeomorphism

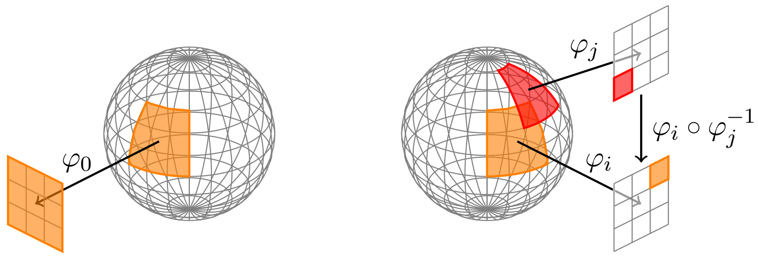
In practice, it is often difficult to check whether a function is continuous. To check differentiability is on the other hand easier. Thus, we would like to extend the idea of homeomorphisms to the class of differentiable functions.

Definition 5. A bijection $\varphi\colon X \to Y$ is called a **C^k-diffeomorphism** iff φ and φ^{-1} are C^k-functions, *i.e.*, for all $i \leqslant k$ exist the i-th derivatives of these functions and they are continuous. C^∞-diffeomorphisms are called **(smooth) diffeomorphisms**.

Problem: While we know how to "extend" the idea of dimensions to $\mathbb{R}^n$ **non-linearly**, we still need to define when $C \subset \mathbb{R}^2$ is a 1D object.

Solution: Concept of **manifolds** and **sub-manifolds**.

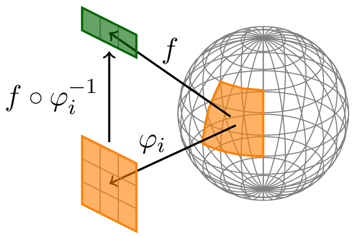
Chart and Atlas



Definition 6. (M_0, φ_0) is called an **n-dimensional chart** of M iff $\varphi_0 \colon M_0 \to U_0$ is a homeomorphism between the open sets $M_0 \subset M$ and $U_0 \subset \mathbb{R}^n$.

Definition 7. A collection $(M_i, \varphi_i)_{i \in \mathcal{I}}$ of charts is called a **C^k atlas** iff $M = \bigcup_{i \in \mathcal{I}} M_i$ and for any two charts φ_i and φ_j, the mapping $\varphi_i \circ \varphi_j^{-1}$ is a C^k-diffeomorphism between $\varphi_j(M_i \cap M_j)$ and $\varphi_i(M_i \cap M_j)$.

Smooth Functions



Definition 8. A set M with a C^∞ atlas $(M_i, \varphi_i)_{i \in \mathcal{I}}$ is called a **(smooth) manifold**.

Definition 9. Given a manifold M, a function $f \colon M \to \mathbb{R}^d$ is called **smooth** iff for all charts (M_i, φ_i), the function $f \circ \varphi_i^{-1} \colon \varphi_i(M_i) \to \mathbb{R}^d$ is smooth.

Whether a function is smooth depends very much on the chosen atlas. Therefore, we are more interested in submanifolds.

Generalization of $\mathbb{R}^n$

We like to think of n-dimensional manifolds as extensions of the linear space $\mathbb{R}^n$. In order to see this we will define the “manifold” $\mathbb{R}^n$.

Let $(M_i)_{i\in\mathcal{I}}$ be a collection of open sets $M_i \subset \mathbb{R}^n$ that cover $\mathbb{R}^n$. Choices are:

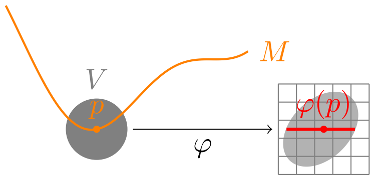
$$
\begin{aligned}
\mathcal{I} &= \mathbb{Z}^n & M_{(i_1,\ldots,i_n)} &= \left\{x \in \mathbb{R}^n \,\middle|\, \|x - i\| < \sqrt{n}\right\} \\
\mathcal{I} &= \{0\} & M_0 &= \mathbb{R}^n
\end{aligned}
$$

Given these sets, the charts can be choosen as (M_i, id_{M_i}) and for overlapping charts, we obtain the diffeomorphism

$$
\begin{aligned}
\varphi_i \circ \varphi_j^{-1} \colon M_i \cap M_j &\to M_i \cap M_j \\
p &\mapsto p
\end{aligned}
$$

With these atlases, we obtain that the “smooth functions” f on the “manifold” $\mathbb{R}^n$ are exactly the functions $C^\infty(\mathbb{R}^n)$.

Smooth Submanifold



Definition 10. A subset $M \subset \mathbb{R}^n$ is a **(smooth) submanifold** of dimension m iff for every point $p \in M$, there exists a chart (V, φ) of $\mathbb{R}^n$ such that

- $p \in V$.
- $\varphi(M \cap V) = \mathbb{R}^m \cap \varphi(V)$.

We call $n - m$ the **co-dimension** of M.

Note that φ is automatically a diffeomorphism. Why?
Note that M is automatically a manifold. Why?

Implicit Function Theorem

Theorem 4 (Implicit Function Theorem). *Given a C^k-mapping $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $(x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $F(x_0, y_0) = 0$ and $\frac{\partial}{\partial y}F(x_0, y_0)$ is invertible.*
Then there exists $U \in \mathcal{T}(\mathbb{R}^m)$, $V \in \mathcal{T}(\mathbb{R}^n)$ and $\varphi\colon U \to V$ such that

- $x_0 \in U$, $y_0 \in V$ *and* $\varphi(x_0) = y_0$.
- *For all* $x \in U$ *we have* $F(x, \varphi(x)) = 0$.
- φ *is a bijective* C^k*-mapping.*
- *For all* $x \in U$ *we have* $\frac{\partial}{\partial x}\varphi(x) = -\left[\frac{\partial}{\partial y}F(x, \varphi(x))\right]^{-1} \frac{\partial}{\partial x}F(x, \varphi(x))$.

Theorem 5 (Inverse Function Theorem). *Given a C^k-mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $Df(x_0)$ is of maximal rank for a $x_0 \in \mathbb{R}^n$. Then there exists $U \in \mathcal{T}(\mathbb{R}^n)$, $V \in \mathcal{T}(\mathbb{R}^n)$ such that $f\colon U \to V$ is a C^k-diffeomorphism and $D\left(f^{-1}\right)(f(x)) = (Df)^{-1}(x)$*

Proof. Define $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $F(x, y) = x - f(y)$ and apply the *Implicit Function Theorem*. □

Explicit Submanifolds

One important example of a submanifold is described by smooth coordinate mappings $x\colon U \to \mathbb{R}^n$:

Lemma 1. *A subset $M \subset \mathbb{R}^n$ together with smooth* **coordinate mappings** $(x_i, U_i)_{i \in \mathcal{I}}$ *is a smooth submanifold of dimension m if the following holds:*

- *All U_i are open subsets of $\mathbb{R}^m$.*
- $M = \bigcup_{i \in \mathcal{I}} x_i(U_i)$.
- *For all $u \in U_i$, x_i is smooth and $Dx_i(u) \in \mathbb{R}^{n \times m}$ is of maximal rank m.*

Proof. Given $p \in M$, we choose $i \in \mathcal{I}$ and $\hat{u} \in \mathbb{R}^m$ such that $p = x_i(\hat{u})$.
Since $Dx_i(\hat{u})$ is of maximal rank, we can find a matrix $A_0 \in \mathbb{R}^{n \times (n-m)}$ such that $A := \begin{pmatrix} Dx_i(\hat{u}) & A_0 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is of maximal rank n.

Explicit Submanifolds

Proof (Cont.) As a result, we can define the smooth function

$$\begin{aligned} \psi : U_i \times \mathbb{R}^{n-m} &\to \mathbb{R}^n \\ (u_1, \dots, u_m, v_1, \dots, v_{n-m}) &\mapsto \varphi(u) + A \cdot v \end{aligned}$$

with $\det(D\psi(\hat{u}, 0)) \neq 0$. Using the *Inverse Function Theorem* proves the lemma. □

Now we can prove that

$$C := \{(x, y) \in \mathbb{R}^2 | y = x^2\}$$

is a manifold of dimension 1. How?

Implicit Submanifolds

In practice, it is often difficult to define different charts or coordinate functions. Instead, we like to define the manifold M by formulating certain constraints, *e.g.*,

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3 | \langle x, x \rangle = 1\}$$

Lemma 2. *Given a function* $f : \mathbb{R}^n \to \mathbb{R}^k$ *and a* **regular value** $c \in \mathbb{R}^k$, i.e.,

$$x \in f^{-1}(c) \quad \Rightarrow \quad \operatorname{rank}(Df(x)) = k.$$

Then, $M := f^{-1}(c)$ *is a submanifold of co-dimension* k.

Implicit Submanifolds

Proof. Let $p \in M \subset \mathbb{R}^n$. Since $Df(p)$ is of rank k, we can find k columns of $Df(p)$ that are linear independent. W.l.o.g. we assume that these k columns are the last k. Thus, the function $f\colon \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies the *Implicit Function Theorem* with respect to $(x_0, y_0) = p$.
The implicit function $\varphi\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ defines a coordinate mapping $x\colon \mathbb{R}^k \to \mathbb{R}^n$ via $x(u) := (u, \varphi(u))$, which proves the lemma.

Note that the implicit submanifold can be transformed into an explicit submanifold.

Given a point $p \in M$ the created coordinate mapping x satisfies

$$Dx(p) = \begin{pmatrix} \mathrm{id} \\ -\left[\frac{\partial f}{\partial y}(p)\right]^{-1} \frac{\partial f}{\partial x}(p) \end{pmatrix}$$

Objects and Shapes

With these definitions in place, we can finally define what we mean by an object and a shape

Definition 11 (Object)**.** An **object** of dimension d is an open subset $X \subset \mathbb{R}^d$ such that its boundary $B := \partial X$ is a submanifold of dimension $d-1$. We will use the notation $\mathcal{O}^d$ for the space of all objects of dimension d.

Some authors only study the boundary of an object.

Definition 12 (Shape)**.** Given an equivalence relation $\sim$ of $\mathcal{O}^d$, the equivalence class $[O]$ of an object $O \in \mathcal{O}^d$ is its shape. We call the set of all shapes the **shape space** $\mathcal{O}^d / \sim$.

For now we will focus on an equivalence relation that uses a combination of rotation, translation and scaling to define equivalent objects.

Tangent Space

Since we want to focus on smooth submanifolds M of dimension m, we like to approximate the direct vicinity of a point $p \in M$ with a linear vector space of dimension m. This leads to the concept of the tangent space.

Definition 13 (Tangent Space). Let $M \subset \mathbb{R}^n$ be a submanifold of dimension $m \leqslant n$ that is given via coordinate functions $(x_i, U_i)_{i \in \mathcal{I}}$. Given $i \in \mathcal{I}$ such that $p = x_i(u)$, we define the **tangent space T_pM** of M at the position p as

$$T_pM := \{Dx_i(u) \cdot v | \forall v \in \mathbb{R}^m\} \qquad \left[= \operatorname{Im}(Dx_i(u))\right]$$

Lemma 3 (Tangent Space). *The tangent space $T_pM \subset \mathbb{R}^n$ is a linear subspace and does not depend on the choice of the choosen coordinate map (x_i, U_i).*

Proof. Excercise. □

Tangent Space

Lemma 4 (Tangent Space). *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function with regular value $c \in \mathbb{R}^k$ and $M := f^{-1}(c)$ the manifold with respect to this value. For every $p \in M$ we have*

$$T_pM := \{v \in \mathbb{R}^n | Df(p) \cdot v = 0\} \qquad \left[= \ker(Df(p))\right]$$

Proof. The coordinate mapping x that we constructed for a implicetly defined manifold is $Dx(u) = \left(\begin{smallmatrix} \mathrm{id} \\ -[\frac{\partial f}{\partial y}(p)]^{-1} \frac{\partial f}{\partial x}(p) \end{smallmatrix}\right)$. Using linear algebra, we obtain

$$\begin{aligned} & y \in T_pM^{\perp} = \operatorname{Im}\left(Dx(u)\right)^{\perp} = \ker\left(Dx(u)^{\top}\right) \\ \Rightarrow & 0 = y_1 - \left[\frac{\partial f}{\partial x}(p)\right]^{\top} \left[\frac{\partial f}{\partial y}(p)\right]^{-\top} y_2 \end{aligned}$$

Choosing $y_2 = [\frac{\partial f}{\partial y}(p)]^{\top} \cdot v$ leads to $y_1 = [\frac{\partial f}{\partial x}(p)]^{\top} \cdot v$, which proves $T_pM = \operatorname{Im}(Df(p)^{\top})^{\perp} = \ker(Df(p))$.

Literature

Dimension

- Peano, *Sur une courbe, qui remplit toute une aire plane*, 1890, Math. Annalen (36), 157–160.
- Brouwer, *Beweis der Invarianz der Dimensionenzahl*, 1911, Math. Annalen (70), 161–165.

Smooth Manifolds

- Poincaré, *Analysis Situs*, 1895, Journal de l'École Polytechnique.