

Analysis of 3D Shapes (IN2238)

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3. Differential and Curvature

Differential

History of Differential

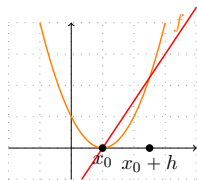
While the concept of the **derivative** or **differential** is nowadays one of the basic concepts in modern mathematics, it took a while to find a clean mathematical definition.

The notation $\frac{dy}{dx}$ is due to **Leibniz** who called **dx** and **dy** an "infinitely small change of" x resp. y .

In 1924, **Courant** mentioned that the idea of the differential as infinite small expression "lacks any meaning" and is therefore "useless".

The modern notion of derivatives and differential is due to **Cauchy** and **Weierstraß**, which we want to revise in order to extend it later to smooth mappings between manifolds.

Derivative according to Cauchy

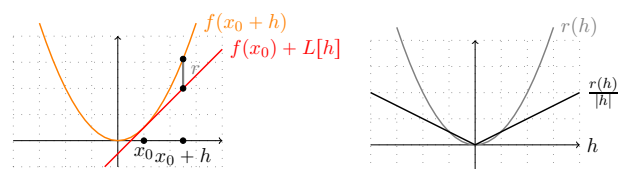


The derivative $f'(x_0)$ of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ at the position $x_0 \in \mathbb{R}$ is

$$f'(x_0) := \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

While this is a working mathematical definition, it is a bit difficult to extend it to arbitrary functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, since we cannot "divide by vectors".

Differential according to Weierstraß



Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$ and a position $x_0 \in \mathbb{R}$, its differential $Df(x_0)$ is the **unique linear mapping** $L: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x_0 + h) = f(x_0) + L[h] + r(h)$$

$$\lim_{h \rightarrow 0} \frac{r(h)}{|h|} = 0$$

Jacobi Matrix

Let $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a differentiable function and $x_0 \in \mathbb{R}^m$. The differential

$$Df(x_0): \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is a linear mapping.

Using the canonical bases $\{e_1, \dots, e_m\}$ for \mathbb{R}^m and $\{e_1, \dots, e_n\}$ for \mathbb{R}^n , $Df(x_0)$ can be written in matrix form, the **Jacobi matrix**

$$Df(x_0)[h] = J \cdot h \quad J = \begin{pmatrix} J_{1,1} & \dots & J_{1,m} \\ \vdots & & \vdots \\ J_{n,1} & \dots & J_{n,m} \end{pmatrix}$$

with

$$J_{i,j} = \langle e_i, J \cdot e_j \rangle = \langle e_i, Df(x_0)[e_j] \rangle = \lim_{h \rightarrow 0} \frac{f^i(x_0 + h \cdot e_j) - f^i(x_0)}{h} = \partial_j f^i(x_0)$$

Chain Rule

Let $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^k \rightarrow \mathbb{R}^m$ be differentiable functions. Then we have

$$\begin{aligned} (f \circ g)(x_0 + h) &= f(g(x_0) + Dg(x_0)[h] + r_g(h)) \\ &= (f \circ g)(x_0) + Df(g(x_0)) [Dg(x_0)[h] + r_g(h)] + \\ &\quad r_f(Dg(x_0)[h] + r_g(h)) \\ &= (f \circ g)(x_0) + Df(g(x_0)) [Dg(x_0)[h]] + r(h) \end{aligned}$$

Thus we have

$$D(f \circ g)(x_0) = Df(g(x_0)) \circ Dg(x_0)$$

Chain Rule (Example)

Differential Push-Forward Curvature of 2D Objects

Let $g_1, g_2: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x, y) := x + y$, we have

$$\begin{aligned} D(g_1 + g_2)(x_0) &= D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\ &= (\text{id} \quad \text{id}) \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} = Dg_1(x_0) + Dg_2(x_0) \end{aligned}$$

Let $g_1, g_2: \mathbb{R}^m \rightarrow \mathbb{R}$ and $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ with $f(x, y) := x \cdot y$, we have

$$\begin{aligned} D(g_1 \cdot g_2)(x_0) &= D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\ &= (g_2(x_0) \quad g_1(x_0)) \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} \\ &= Dg_1(x_0) \cdot g_2(x_0) + Dg_2(x_0) \cdot g_1(x_0) \end{aligned}$$

Interpretation of the Differential

Differential Push-Forward Curvature of 2D Objects

Given a function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and a position $p \in \mathbb{R}^m$, the equation

$$f(p + v) = f(p) + Df(p)[v] + r(v)$$

can be interpreted as following:

- p describes a **point** in the space on which f is defined,
- v describes the **direction** in which we change the point p
- $Df(p)[v]$ describes the **direction** in which f changes if we change the point p in the direction v .

For vector spaces, there is no distinction between points and directions. For manifolds M , points p will be on the manifold and directions on the tangent space $T_p M$.

Push-Forward

Differential Push-Forward Curvature of 2D Objects

Curve Representation of Tangent Vectors

Differential Push-Forward Curvature of 2D Objects

Given a point $p \in M$ of a d -dimensional submanifold $M \subset \mathbb{R}^n$, we can represent a tangent vector $v \in T_p M$ as a curve $c: (-\varepsilon, \varepsilon) \rightarrow M$ with $c(0) = p$.

To see this, let us look at the manifold from the point of view of a coordinate mapping $x: U \rightarrow M$ with $0 \in U \subset \mathbb{R}^d$ and $x(0) = p$.

Since $v \in T_p M = \text{Im}(Dx(0))$, we know that there is an $h \in \mathbb{R}^d$ such that $Dx(0)[h] = v$.

Using

$$c: (-\varepsilon, \varepsilon) \rightarrow M \quad c(t) = x(t \cdot h),$$

we have

$$Dc(0) = Dx(0 \cdot h) \cdot h = v.$$

Alternative Definition of Tangent Vectors

Differential Push-Forward Curvature of 2D Objects

Given a point $p \in M$ of a d -dimensional submanifold $M \subset \mathbb{R}^n$, we define

$$C_p M := \{c: (-\varepsilon, \varepsilon) \rightarrow M \mid \exists \varepsilon > 0 : c \text{ is smooth and } c(0) = p\}.$$

The goal is to define $T_p M$ by defining an equivalence relation on $C_p M$:

$$c_1 \sim c_2 \quad :\Leftrightarrow \quad Dc_1(0) = Dc_2(0),$$

It is easy to check that \sim satisfies *reflexivity*, *symmetry* and *transitivity*.

It turns out $T_p M = C_p M / \sim$, which provides us with an alternative definition for the tangent space $T_p M$.

The advantage of this rather theoretical definition is that for any $v \in T_p M$ we can choose a curve $c \in v$ that passes through p and vice versa, i.e., any curve c that passes through a point p defines a tangent vector $v := [c]$.

Differential as Push-Forward

Differential Push-Forward Curvature of 2D Objects

Given two submanifolds M and N as well as a function $f: M \rightarrow N$. For $p \in M$ and $q = f(p) \in N$, the differential $Df(p)$ is the **push-forward**

$$\begin{aligned} Df(p): T_p M &\rightarrow T_q N \\ [c] &\mapsto [f \circ c] \end{aligned}$$

Assuming that $x_p: U_p \rightarrow M$ is a coordinate mapping for $p = x_p(0)$ and $x_q: U_q \rightarrow N$ is a coordinate mapping for $q = x_q(0)$, the push-forward definition becomes

$$Df(p)[v] = \left. \frac{\partial}{\partial t} (x_q^{-1} \circ f \circ x_p)(t \cdot h) \right|_{t=0},$$

where $v = Dx_p(0)[h]$.

It is easy to show that the push-forward is a linear mapping. Exercise.

Coordinate Interpretation

Differential Push-Forward Curvature of 2D Objects

Given coordinate mappings $x_p: U_p \rightarrow M$ and $x_q: U_q \rightarrow N$ with $p = x_p(0)$ and $q = f(p) = x_q(0)$, the differential becomes

$$Df(p)[v] = \left. \frac{\partial}{\partial t} (x_q^{-1} \circ f \circ x_p)(t \cdot h) \right|_{t=0}.$$

If we were to apply the chain rule, we would obtain

$$Df(p)[v] = \mathbf{D}(x_q^{-1})(q) \cdot \mathbf{D}f(p) \cdot \mathbf{D}x_p(0) \cdot \mathbf{h}$$

- $Dx_p(0) \cdot h$ defines the tangent vector $v \in T_p M$.
- $Df(p)$ is the differential of f ignoring the submanifolds M and N .
- $D(x_q^{-1})(q)$ projects a vector onto $T_q M$.

What is a Matrix?

Differential Push-Forward Curvature of 2D Objects

Linear mappings are commonly represented by matrices. We want to emphasize the difference between a matrix and a linear mapping.

Given an m -dimensional \mathbb{R} -vector space X , an n -dimensional \mathbb{R} -vector space Y and a linear mapping $L: X \rightarrow Y$, we can represent L by finite many scalars.

To this end, let $\mathcal{B}_X = \{x_1, \dots, x_m\}$ and $\mathcal{B}_Y = \{y_1, \dots, y_n\}$ bases of X and Y respectively. Then we know that for each $x_j \in \mathcal{B}_X$ we have

$$L(x_j) = \sum_{i=1}^n a_{ij} y_i.$$

for some $a_{ij} \in \mathbb{R}$.

We write this a_{ij} in a matrix A and call $A = \mathcal{M}_{\mathcal{B}_Y}^{\mathcal{B}_X}(L) \in \mathbb{R}^{n \times m}$ the representing matrix of L with respect to the basis \mathcal{B}_X and \mathcal{B}_Y .

Given two submanifolds M and N as well as a function $f: M \rightarrow N$. For $p \in M$ and $q = f(p) \in N$, the differential $Df(p): T_pM \rightarrow T_qN$ is a linear mapping, but in general we do not have a canonical matrix representation.

This means that any basis \mathcal{B}_p of T_pM and \mathcal{B}_q of T_qN would define a different matrix $\mathcal{M}_{\mathcal{B}_q}^{\mathcal{B}_p}(Df(p)) \in \mathbb{R}^{n \times m}$ with $n = \dim(N)$ and $m = \dim(M)$.

Since $T_pM = \text{Im}(Dx(0))$, $\mathcal{B}_p = \{Dx(0)[e_1], \dots, Dx(0)[e_m]\}$ would be a natural way of defining a basis of T_pM . Nonetheless, the resulting matrix would then depend on the coordinate mappings x_p and x_q that we choose for $p \in M$ and $q \in N$ respectively.

While there is no unique matrix that describes the differential, it is important to note that $\text{rank } Df(p)$ is independent of the chosen coordinate mappings.

Curvature of 2D Objects

Planar Curves and Normals

Given a 2D object O and its boundary, the 1D submanifold $M := \partial O$, we like to define the normal vector $n(p)$ for each point $p \in M$.

Given a coordinate mapping $x: U \rightarrow M$ with $x(0) = p$, we have $T_pM = \text{Im}(Dx(0))$ and a normal vector might be defined via

$$n(p) = \frac{1}{\|Dx(0)\|} \begin{pmatrix} +Dx^2(0) \\ -Dx^1(0) \end{pmatrix} \in \mathbb{S}^1$$

Since M is of codimension 1, $n(p)$ is up to the sign uniquely defined.

Thus, we have a smooth mapping

$$n: M \rightarrow \mathbb{S}^1$$

that defines a unique normal vector field of M . Why?

Differential of the Normal Mapping

Given a point $p \in M$ of $M \subset \mathbb{R}^2$, we have

$$Dn(p): T_pM \rightarrow T_{n(p)}\mathbb{S}^1.$$

Since we have

$$T_{n(p)}\mathbb{S}^1 = n(p)^\perp = T_pM,$$

we know that $Dn(p)$ is an **endomorphism**, i.e., a linear mapping that maps the vector space T_pM onto itself.

Because $\dim T_pM = 1$, $Dn(p)$ maps a vector $v \in T_pM$ to $\kappa(p) \cdot v$.

This scalar value $\kappa(p) \in \mathbb{R}$ is called the **curvature** of M at the position p .

Derivative of the Normal Field

If we take the derivative of $N = (N^1, N^2): U \rightarrow \mathbb{R}^2$, $t \mapsto n \circ x(t)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} N^1(x(t)) &= \frac{\partial \dot{x}^2(t)}{\partial t \|\dot{x}(t)\|} = \frac{\ddot{x}^2(t) \|\dot{x}(t)\| - \dot{x}^2(t) \frac{\dot{x}^1(t) + \dot{x}^2(t)}{\|\dot{x}(t)\|}}{\|\dot{x}(t)\|^2} \\ &= \frac{\ddot{x}^2(t) \|\dot{x}(t)\|^2 - \dot{x}^2(t) \cdot (\dot{x}^1(t) + \dot{x}^2(t))}{\|\dot{x}(t)\|^3} \\ \frac{\partial}{\partial t} N^2(x(t)) &= \frac{-\dot{x}^1(t) \|\dot{x}(t)\|^2 + \dot{x}^1(t) \cdot (\dot{x}^1(t) + \dot{x}^2(t))}{\|\dot{x}(t)\|^3} \end{aligned}$$

Note that $DN(p)$ is not necessarily in T_pM . Thus, we have to project it onto T_pM . To this end, let us choose $\{\dot{x}(t)\}$ as the base of T_pM .

Curvature

Overall, we have $Dn(p)[\dot{x}(t)] = \kappa(t) \cdot \dot{x}(t)$.

Therefore, we have

$$\begin{aligned} \kappa(t) &= \frac{\langle \dot{N}(t), \dot{x}(t) \rangle}{\|\dot{x}(t)\|^2} = \frac{\dot{x}^1(t)\dot{x}^2(t)\|\dot{x}(t)\|^2 - \dot{x}^2(t)\dot{x}^1(t)\|\dot{x}(t)\|^2}{\|\dot{x}(t)\|^5} \\ &= \frac{\det(\dot{x}(t), \ddot{x}(t))}{\|\dot{x}(t)\|^3} \end{aligned}$$

By construction, we know that curvature is invariant with respect to

- Translation. Why?
- Rotation. Why?
- Reparametrization. Why?

Curvature of Implicit Submanifolds

If $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ has the regular value $c \in \mathbb{R}$, how can we use F in order to compute the curvature of M at $p \in M$?

The normal field n can be defined as $n(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}$.

Since n is also defined in a neighborhood of M , we can compute its derivative $Dn: M \rightarrow \mathbb{R}^{2 \times 2}$. If we write the linear mapping $Dn(p)$ with respect to the basis $\mathcal{B}_p = \{\nabla F(p), \nabla F(p)^\perp\}$, we obtain

$$\mathcal{M}_{\mathcal{B}_p}^{\mathcal{B}_p}(Dn(p)) = \begin{pmatrix} 0 & * \\ * & \kappa(p) \end{pmatrix}.$$

Therefore, we have

$$\kappa(p) = \text{tr } Dn(p) = \text{div} \left(\frac{\nabla F(p)}{\|\nabla F(p)\|} \right)$$

Literature

Differential

- Cauchy, *Cours d'Analyse de l'École Royale Polytechnique; I.re Partie. Analyse algébrique*, 1821.