Analysis of 3D Shapes (IN2238)

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Summer Semester 2016

3. Differential and Curvature

Differential

History of Differential

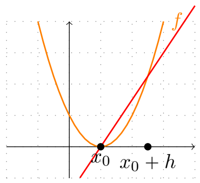
While the concept of the **derivative** or **differential** is nowadays one of the basic concepts in modern mathematics, it took a while to find a clean mathematical definition.

The notation $\frac{\mathrm{dy}}{\mathrm{dx}}$ is due to **Leibniz** who called **dx** and **dy** an "infinitely small change of" x resp. y.

In 1924, **Courant** mentioned that the idea of the differential as infinite small expression "lacks any meaning" and is therefore "useless".

The modern notion of derivatives and differential is due to **Cauchy** and **Weierstraß**, which we want to revise in order to extend it later to smooth mappings between manifolds.

Derivative according to Cauchy

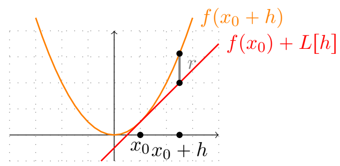


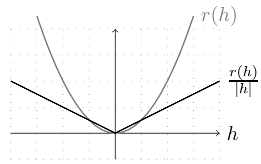
The derivative $f'(x_0)$ of a function $f\colon \mathbb{R} \to \mathbb{R}$ at the position $x_0 \in \mathbb{R}$ is

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

While this is a working mathematical definition, it is a bit difficult to extend it to arbitrary functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$, since we cannot "divide by vectors".

Differential according to Weierstraß





Given a function $f\colon \mathbb{R} \to \mathbb{R}$ and a postion $x_0 \in \mathbb{R}$, its differential $Df(x_0)$ is the **unique linear mapping** $L\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} f(x_0 + h) &= f(x_0) + L[h] + r(h) \\ \lim_{h \to 0} \frac{r(h)}{|h|} &= 0 \end{aligned}$$

Jacobi Matrix

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function and $x_0 \in \mathbb{R}^m$. The differential

$$Df(x_0)\colon \mathbb{R}^m \to \mathbb{R}^n$$

is a linear mapping.

Using the canonical bases $\{e_1, \ldots, e_m\}$ for $\mathbb{R}^m$ and $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$, $Df(x_0)$ can be written in matrix form, the **Jacobi matrix**

$$Df(x_0)[h] = J \cdot h \qquad J = \begin{pmatrix} J_{1,1} & \cdots & J_{1,m} \\ \vdots & & \vdots \\ J_{n,1} & \cdots & J_{n,m} \end{pmatrix}$$

with

$$J_{i,j} = \langle e_i, J \cdot e_j \rangle = \langle e_i, Df(x_0)[e_j] \rangle = \lim_{h \to 0} \frac{f^i(x_0 + h \cdot e_j) - f^i(x_0)}{h} = \partial_j f^i(x_0)$$

Chain Rule

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^k \to \mathbb{R}^m$ be differentiable functions. Then we have

$$\begin{aligned} (f \circ g)(x_0 + h) =& f\left(g(x_0) + Dg(x_0)[h] + r_g(h)\right) \\ =& (f \circ g)(x_0) + Df(g(x_0))\left[Dg(x_0)[h] + r_g(h)\right] + \\ & r_f\left(Dg(x_0)[h] + r_g(h)\right) \\ =& (f \circ g)(x_0) + Df(g(x_0))\left[Dg(x_0)[h]\right] + r(h) \end{aligned}$$

Thus we have

$$D(\mathbf{f} \circ \mathbf{g})(x_0) = \mathbf{Df}(g(x_0)) \circ \mathbf{Dg}(x_0)$$

Chain Rule (Example)

Let $g_1, g_2 \colon \mathbb{R}^m \to \mathbb{R}^n$ and $f \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(x,y) := x + y$, we have

$$
\begin{aligned}
D(g_1 + g_2)(x_0) =& D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\
=& \begin{pmatrix} \mathrm{id} & \mathrm{id} \end{pmatrix} \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} = Dg_1(x_0) + Dg_2(x_0)
\end{aligned}
$$

Let $g_1, g_2 \colon \mathbb{R}^m \to \mathbb{R}$ and $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $f(x,y) := x \cdot y$, we have

$$
\begin{aligned}
D(g_1 \cdot g_2)(x_0) =& D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\
=& \begin{pmatrix} g_2(x_0) & g_1(x_0) \end{pmatrix} \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} \\
=& Dg_1(x_0) \cdot g_2(x_0) + Dg_2(x_0) \cdot g_1(x_0)
\end{aligned}
$$

Interpretation of the Differential

Given a function $f \colon \mathbb{R}^m \to \mathbb{R}^n$ and a position $p \in \mathbb{R}^m$, the equation

$$
f(p + v) = f(p) + Df(p)[v] + r(v)
$$

can be interpreted as following:

- p describes a **point** in the space on which f is defined,
- v describes the **direction** in which we change the point p
- $Df(p)[v]$ describes the **direction** in which f changes if we change the point p in the direction v.

For vector spaces, there is no distinction between points and directions. For manifolds M, points p will be on the manifold and directions on the tangent space T_pM.

Push-Forward

Curve Representation of Tangent Vectors

Given a point $p \in M$ of a d-dimensional submanifold $M \subset \mathbb{R}^n$, we can represent a tangent vector $v \in T_pM$ as a curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$.

To see this, let us look at the manifold from the point of view of a coordinate mapping $x \colon U \to M$ with $0 \in U \subset \mathbb{R}^d$ and $x(0) = p$.

Since $v \in T_pM = \mathrm{Im}(Dx(0))$, we know that there is an $h \in \mathbb{R}^d$ such that $Dx(0)[h] = v$.

Using

$$
c \colon (-\varepsilon, \varepsilon) \to M \qquad\qquad c(t) = x\,(t \cdot h)\,,
$$

we have

$$
Dc(0) = Dx(0 \cdot h) \cdot h = v.
$$

Alternative Definition of Tangent Vectors

Given a point $p \in M$ of a d-dimensional submanifold $M \subset \mathbb{R}^n$, we define

$$
\mathcal{C}_pM := \{c \colon (-\varepsilon, \varepsilon) \to M | \exists \varepsilon > 0 : c \text{ is smooth and } c(0) = p\}.
$$

The goal is to define T_pM by defining an equivalence relation on $\mathcal{C}_pM$:

$$
c_1 \sim c_2 \qquad\qquad :\Leftrightarrow \qquad\qquad Dc_1(0) = Dc_2(0),
$$

It is easy to check that $\sim$ satisfies *reflexivity*, *symmetry* and *transitivity*.

It turns out $T_pM = \mathcal{C}_pM / \sim$, which provides us with an alternative definition for the tangent space T_pM.

The advantage of this rather theoretical definition is that for any $v \in T_pM$ we can choose a curve $c \in v$ that passes through p and vice versa, *i.e.*, any curve c that passes through a point p defines a tangent vector $v := [c]$.

Differential as Push-Forward

Given two submanifolds M and N as well as a function $f \colon M \to N$. For $p \in M$ and $q = f(p) \in N$, the differential $Df(p)$ is the **push-forward**

$$
\begin{aligned}
Df(p) \colon T_pM \to& T_qN \\
[c] \mapsto& [f \circ c]
\end{aligned}
$$

Assuming that $x_p \colon U_p \to M$ is a coordinate mapping for $p = x_p(0)$ and $x_q \colon U_q \to N$ is a coordinate mapping for $q = x_q(0)$, the push-forward definition becomes

$$
Df(p)[v] = \frac{\partial}{\partial t}(x_q \circ x_q^{-1} \circ f \circ x_p)(t \cdot h)\bigg|_{t=0},
$$

where $v = Dx_p(0)[h]$.

It is easy to show that the push-forward is a linear mapping. Excercise.

Coordinate Interpretation

Given coordinate mappings $x_p \colon U_p \to M$ and $x_q \colon U_q \to N$ with $p = x_p(0)$ and $q = f(p) = x_q(0)$, the differential becomes

$$
Df(p)[v] = \frac{\partial}{\partial t}(x_q \circ x_q^{-1} \circ f \circ x_p)(t \cdot h)\bigg|_{t=0}.
$$

If we were to apply the chain rule, we would obtain

$$
Df(p)[v] = \mathbf{D(x_q)}(x^{-1}(q)) \cdot \mathbf{D(x_q^{-1})}(q) \cdot \mathbf{Df}(p) \cdot \mathbf{Dx_p}(0) \cdot \mathbf{h}
$$

- $Dx_p(0) \cdot h$ defines the tangent vector $v \in T_pM$.
- $Df(p)$ is the differential of f ignoring the submanifolds M and N.
- $D(x_q^{-1})(q)$ is the pseudo-inverse of $D(x_q)(x_q^{-1}(q))$.
- $D(x_q)(x^{-1}(q)) \cdot D(x_q^{-1})(q)$ projects a vector onto T_qM.

What is a Matrix?

Linear mappings are commonly represented by matrices. We want to emphasize the difference between a matrix and a linear mapping.

Given an m-dimensional $\mathbb{R}$-vector space X, an n-dimensional $\mathbb{R}$-vector space Y and a linear mapping $L \colon X \to Y$, we can represent L by finite many scalars.

To this end, let $\mathcal{B}_X = \{x_1, \ldots, x_m\}$ and $\mathcal{B}_Y = \{y_1, \ldots, y_n\}$ bases of X and Y respectively. Then we know that for each $x_j \in \mathcal{B}_X$ we have

$$
L(x_j) = \sum_{i=1}^{n} a_{ij} y_i.
$$

for some $a_{ij} \in \mathbb{R}$.

We write this a_{ij} in a matrix A and call $A = \mathcal{M}_{\mathcal{B}_Y}^{\mathcal{B}_X}(L) \in \mathbb{R}^{n \times m}$ the representing matrix of L with respect to the basis $\mathcal{B}_X$ and $\mathcal{B}_Y$.

Matrix of the Differential

Given two submanifolds M and N as well as a function $f\colon M \to N$. For $p \in M$ and $q = f(p) \in N$, the differential $Df(p)\colon T_pM \to T_qN$ is a linear mapping, but in general we do not have a canonical matrix representation.

This means that any basis $\mathcal{B}_p$ of T_pM and $\mathcal{B}_q$ of T_qN would define a different matrix $\mathcal{M}_{\mathcal{B}_q}^{\mathcal{B}_p}(Df(p)) \in \mathbb{R}^{n\times m}$ with $n = \dim(N)$ and $m = \dim(M)$.

Since $T_pM = \operatorname{Im}(Dx(0))$, $\mathcal{B}_p = \{Dx(0)[e_1], \ldots, Dx(0)[e_m]\}$ would be a natural way of defining a basis of T_pM. Nonetheless, the resulting matrix would then depend on the coordinate mappings x_p and x_q that we choose for $p \in M$ and $q \in N$ respectively.

While there is no unique matrix that describes the differential, it is important to note that $\operatorname{rank} Df(p)$ is independent of the choosen coordinate mappings.

Curvature of 2D Objects

Planar Curves and Normals

Given a 2D object O and its boundary, the 1D submanifold $M := \partial O$, we like to define the normal vector $n(p)$ for each point $p \in M$.

Given a coordinate mapping $x\colon U \to M$ with $x(0) = p$, we have $T_pM = \operatorname{Im}(Dx(0))$ and a normal vector might be defined via

$$n(p) = \frac{1}{\|Dx(0)\|}\begin{pmatrix} +Dx^2(0) \\ -Dx^1(0) \end{pmatrix} \in \mathbb{S}^1$$

Since M is of codimension 1, $n(p)$ is up to the sign uniquely defined.

Thus, we have a smooth mapping

$$n\colon M \to \mathbb{S}^1$$

that defines a unique normal vector field of M. Why?

Differential of the Normal Mapping

Given a point $p \in M$ of $M \subset \mathbb{R}^2$, we have

$$Dn(p)\colon T_pM \to T_{n(p)}\mathbb{S}^1.$$

Since we have

$$T_{n(p)}\mathbb{S}^1 = n(p)^\perp = T_pM,$$

we know that $Dn(p)$ is an **endomorphism**, *i.e.*, a linear mapping that maps the vector space T_pM onto itself.

Because $\dim T_pM = 1$, $Dn(p)$ maps a vector $v \in T_pM$ to $\kappa(p) \cdot v$.

This scalar value $\kappa(p) \in \mathbb{R}$ is called the **curvature** of M at the position p.

Derivative of the Normal Field

If we take the derivative of $N = (N^1, N^2)\colon U \to \mathbb{R}^2$, $t \mapsto n \circ x(t)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}N^1(x(t)) &= \frac{\partial}{\partial t}\frac{\dot{x}^2(t)}{\|\dot{x}(t)\|} = \frac{\ddot{x}^2(t)\,\|\dot{x}(t)\| - \dot{x}^2(t)\frac{\ddot{x}^1(t)+\ddot{x}^2(t)}{\|\dot{x}(t)\|}}{\|\dot{x}(t)\|^2} \\
&= \frac{\ddot{x}^2(t)\,\|\dot{x}(t)\|^2 - \dot{x}^2(t)\cdot(\ddot{x}^1(t) + \ddot{x}^2(t))}{\|\dot{x}(t)\|^3} \\
\frac{\partial}{\partial t}N^2(x(t)) &= \frac{-\ddot{x}^1(t)\,\|\dot{x}(t)\|^2 + \dot{x}^1(t)\cdot(\ddot{x}^1(t) + \ddot{x}^2(t))}{\|\dot{x}(t)\|^3}
\end{aligned}$$

Note that $DN(p)$ is not necessarily in T_pM. Thus, we have to project it onto T_pM. To this end, let us choose $\{\dot{x}(t)\}$ as the base of T_pM.

Curvature

Overall, we have $Dn(p)[\dot{x}(t)] = \kappa(t) \cdot \dot{x}(t)$.

Therefore, we have

$$\begin{aligned}
\kappa(t) &= \frac{\left\langle \dot{N}(t), \dot{x}(t) \right\rangle}{\|\dot{x}(t)\|^2} = \frac{\dot{x}^1(t)\ddot{x}^2(t)\,\|\dot{x}(t)\|^2 - \dot{x}^2(t)\ddot{x}^1(t)\,\|\dot{x}(t)\|^2}{\|\dot{x}(t)\|^5} \\
&= \frac{\det\left(\dot{x}(t), \ddot{x}(t)\right)}{\|\dot{x}(t)\|^3}
\end{aligned}$$

By construction, we know that curvature is invariant with respect to

- Translation. Why?
- Rotation. Why?
- Reparametrization. Why?

Curvature of Implicit Submanifolds

If $F\colon \mathbb{R}^2 \to \mathbb{R}$ has the regular value $c \in \mathbb{R}$, how can we use F in order to compute the curvature of M at $p \in M$?

The normal field n can be defined as $n(p) = \frac{\nabla F(p)}{\|\nabla F(p)\|}$.

Since n is also defined in a neighborhood of M, we can compute its derivative $Dn\colon M \to \mathbb{R}^{2\times 2}$. If we write the linear mapping $Dn(p)$ with respect to the basis $\mathcal{B}_p = \{\nabla F(p), \nabla F(p)^\perp\}$, we obtain

$$\mathcal{M}_{\mathcal{B}_p}^{\mathcal{B}_p}(Dn(p)) = \begin{pmatrix} 0 & * \\ * & \kappa(p) \end{pmatrix}.$$

Therefore, we have

$$\kappa(p) = \operatorname{tr} Dn(p) = \operatorname{div}\left(\frac{\nabla F(p)}{\|\nabla F(p)\|}\right)$$

Literature

Differential

- Cauchy, *Cours d'Analyse de l'École Royale Polytechnique; I.re Partie. Analyse algébrique*, 1821.