Analysis of 3D Shapes (IN2238)

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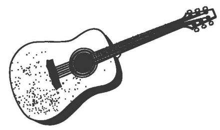
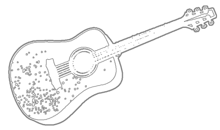
4. Feature Representation and Linear Assignment Problem

Curves

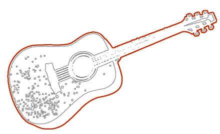
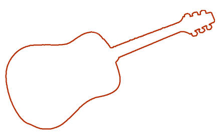
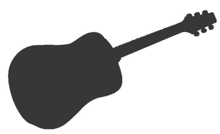
2D Objects

A 2D object is an open set $O \subset \mathbb{R}^2$ such that $B := \partial O$ is a submanifold of dimension 1.

A result from differential geometry is that a 1D manifold is either homeomorphic to $\mathbb{S}^1$ or to $\mathbb{R}$. Since we want to represent an object in a compact image domain $\Omega \subset \mathbb{R}^2$, we can assume that B is a collection of closed contours (each homeomorphic to $\mathbb{S}^1$).

Outer Contour

Assuming that $B = \partial O = \bigcup_{i=1}^k C_i$ is the union of disjoint contours C_i, it is often enough to consider only the **outer contour** of B.

This is equivalent of considering a slightly different object $O' \supset O$ that perceptially is very similar to the original object O.

In conclusion, we assume that $C = \partial O$ is a connected submanifold of dimension 1 that is diffeomorphic to $\mathbb{S}^1$. That means we have

$$c\colon \mathbb{S}^1 \to \mathbb{R}^2 \qquad \|\dot c(t)\| \neq 0 \quad (\forall t \in \mathbb{S}^1).$$

Contour Length

Given a curve $c\colon \mathbb{S}^1 \to \mathbb{R}^2$, its **length** is

$$\begin{aligned}\text{length}(c) &= \lim_{N\to\infty} \sum_{k=1}^{N} \left\| c\left(e^{\frac{2\pi k}{N} i}\right) - c\left(e^{\frac{2\pi(k-1)}{N} i}\right)\right\| \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \left\| \frac{c\left(e^{\frac{2\pi k}{N} i}\right) - c\left(e^{\frac{2\pi(k-1)}{N} i}\right)}{\frac{2\pi}{N}}\right\| \cdot \frac{2\pi}{N} \\ &= \int_{\mathbb{S}^1} \|Dc(t)[t\cdot i]\|\,\mathrm{d}t = \int_{\mathbb{S}^1} \|\dot c(t)\|\,\mathrm{d}t\end{aligned}$$

We call c a **uniform parametrization** of $C = \text{Im}(c)$ iff $\|\dot c(t)\|$ is constant.
Iff this constant is 1, we call c the **arclength parametrization** of C.

Uniform Parametrization

To every curve $c\colon \mathbb{S}^1 \to \mathbb{R}^2$ of C, we can find a different curve that is parametrized uniformly.

To this end let $L := \text{length}(c)$ and

$$\ell\colon [0, 2\pi] \to [0, 2\pi] \qquad \ell(t) = \frac{2\pi}{L} \cdot \int_0^t \left\|\dot c\left(e^{\tau\cdot i}\right)\right\| \mathrm{d}\tau$$

The curve $\hat c\colon \mathbb{S}^1 \to \mathbb{R}^2$ with $\hat c\left(e^{t\cdot i}\right) = c\left(e^{\ell^{-1}(t)\cdot i}\right)$ satisfies

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\hat c\left(e^{t\cdot i}\right)\right\| &= \left\|Dc\left(e^{\ell^{-1}(t)\cdot i}\right)\left[e^{\ell^{-1}(t)\cdot i}\cdot i\right] \cdot \left\|\dot c\left(e^{\ell^{-1}(t)\cdot i}\right)\right\|^{-1}\right\| \frac{L}{2\pi} \\ &= \frac{L}{2\pi}\end{aligned}$$

Curvature

For every uniformly parametrized curve $c\colon \mathbb{S}^1 \to \mathbb{R}^2$, the expression to compute the curvature can be simplified.

Since we have that $\langle \dot c(t), \dot c(t)\rangle$ is constant in t, we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \dot c(t), \dot c(t)\rangle = 2\langle \ddot c(t), \dot c(t)\rangle$$

Thus $\dot c(t)$ and $\ddot c(t)$ are orthogonal to one another and

$$\det(\dot c(t), \ddot c(t)) = \pm\|\dot c(t)\| \cdot \|\ddot c(t)\| = \pm\frac{L}{2\pi}\|\ddot c(t)\|.$$

Therefore, we have for the curvature $\kappa(c(t))$

$$|\kappa(c(t))| = \left|\frac{\det(\dot c(t), \ddot c(t))}{\|\dot c(t)\|^3}\right| = \|\ddot c(t)\| \frac{4\pi^2}{L^2}$$

Shape Matching

Curvature and Shapes

We already saw that the curvature is invariant with respect to translation and rotation.

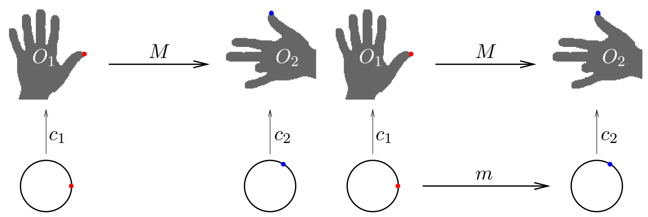
Therefore, we can interpret the curvature mapping $\kappa\colon \mathbb{S}^1 \to \mathbb{R}$ as a **shape representation**.

While we excluded the flexibility with respect to translation and rotation, the shape representation via curvature is not unique.

By using an arbitrary self mapping $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$, we change the curve and the curvature representation

$$\begin{aligned} c\colon \mathbb{S}^1 \to \mathbb{R}^2 \quad &\rightsquigarrow \quad c \circ \varphi\colon \mathbb{S}^1 \to \mathbb{R}^2 \\ \kappa\colon \mathbb{S}^1 \to \mathbb{R} \quad &\rightsquigarrow \quad \kappa \circ \varphi\colon \mathbb{S}^1 \to \mathbb{R} \end{aligned}$$

Shape Matching



A **shape matching** is a mapping $M\colon \partial O_1 \to \partial O_2$ that mapps corresponding boundary points onto one another.

It is easier to define a matching between the parametrization domains of both contours, resulting in $m\colon \mathbb{S}^1 \to \mathbb{S}^1$.

Feature Representation of 2D Shapes

To perform shape matching, we need a **shape feature** that describes the "shapeness" of a curve rather than the curve itself. In the last decades a lot of descriptive shape feature have been developed.

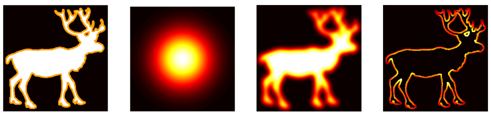
Definition 1. Let $\sim$ be the equivalence relation of objects that defines a shape. If we can find for each curve $c\colon \mathbb{S}^1 \to \mathbb{R}^2$ a mapping $f_c\colon \mathbb{S}^1 \to \mathbb{R}^k$ such that

$$f_c(t) = f_{c'}(t) \qquad \forall c' \sim c \text{ and } \forall t \in \mathbb{S}^1,$$

we call f_c **a shape feature representation** of c and $\mathbb{R}^k$ its **feature space**.

So far, we showed that **curvature** is a **one-dimensional shape feature** with respect to the shape defined by **translation and rotation**.

Integral Invariant



Object O Kernel φ $\varphi * \mathbb{1}_O$ Feature

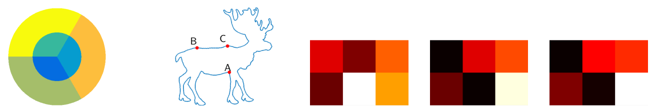
Other shape features like the "integral invariant" will not simply rely on the boundary C of an object O but also on the object itself.
Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a rotation-invariant kernel with compact support, *i.e.*,

$$\begin{aligned} \varphi(x) &= \varphi(R \cdot x) && \forall x \in \mathbb{R} \text{ and } R \in \mathrm{SO}(2) \\ \varphi(x) &= 0 && \forall x \notin B_\varepsilon(0). \end{aligned}$$

Then, we can define the **integral invariant** via the following convolution

$$f\colon \mathbb{S}^1 \to \mathbb{R} \qquad t \mapsto \int_O \varphi(c(t) - x)\,\mathrm{d}x = (\varphi * \mathbb{1}_O)\,(c(t))$$

Shape Context



Polar-Log Histogram 3 shape points Shape Contexts

The shape context can be seen as an extension of the integral invariants. Instead of one, we use multiple kernels $\varphi_i\colon \mathbb{R}^2 \to \mathbb{R}$ in a log-polar scale. The resulting feature is a high-dimensional histogram representation.

The resulting feature is only translation invariant. To make it rotational invariant, one might use the tangent space at $p \in C$ as a baseline. To make the computation practically feasible, only those rotations are used that are represented by the histogram kernels.

Comparing Features

Given two curves $c_1, c_2\colon \mathbb{S}^1 \to \mathbb{R}^k$ of the same shape together with their shape feature representations $f_1, f_2\colon \mathbb{S}^1 \to \mathbb{R}^k$. If the two points $c_1(t_1)$ and $c_2(t_2)$ correspond to one another, we know that $f_1(t_1) = f_2(t_2)$.

Therefore, we can measure the similarity of two arbitrary points $c_1(t_1)$ and $c_2(t_2)$ via $\mathrm{dist}(f_1(t_2), f_2(t_2))$, where the **distance function** $\mathrm{dist}\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}_0^+$ measures the similarity of two features in $\mathbb{R}^k$.

Common distance functions are

$$\begin{aligned} \mathrm{dist}(\kappa_1, \kappa_2) &= (\kappa_1 - \kappa_2)^2 && \text{(Curvature)} \\ \mathrm{dist}(I_1, I_2) &= (I_1 - I_2)^2 && \text{(Integral Invariant)} \\ \mathrm{dist}(C^{(1)}, C^{(2)}) &= \sum_{i=1}^{k} \frac{\left(C^{(1)} - C^{(2)}\right)^2}{C^{(1)} + C^{(2)}} && \text{(Shape Context)} \end{aligned}$$

Discretization

In order to solve the shape matching problem, we like to work with a **finite** representation. The process of transforming a continuous problem into such a "finite" problem is called **discretization**.

Let us assume that two curves $c_1, c_2\colon \mathbb{S}^1 \to \mathbb{R}^2$ are provided in a uniform parametrization. Given the corresponding features $f_1, f_2\colon \mathbb{S}^1 \to \mathbb{R}^k$, we choose the following discretization

$$\begin{aligned} F^{(1)} &= \left(f_1\left(e^{\frac{2\pi}{N}} \cdot i\right) \quad \cdots \quad f_1\left(e^{\frac{2\pi \cdot j}{N}} \cdot i\right) \quad \cdots \quad f_1\left(e^{\frac{2\pi \cdot N}{N}} \cdot i\right)\right) \in \mathbb{R}^{k \times N} \\ F^{(2)} &= \left(f_2\left(e^{\frac{2\pi}{N}} \cdot i\right) \quad \cdots \quad f_2\left(e^{\frac{2\pi \cdot j}{N}} \cdot i\right) \quad \cdots \quad f_2\left(e^{\frac{2\pi \cdot N}{N}} \cdot i\right)\right) \in \mathbb{R}^{k \times N} \end{aligned}$$

This provides us with a **cost matrix** $D \in \mathbb{R}^{N \times N}$, *i.e.*, $D_{i,j} = \mathrm{dist}(F_i^{(1)}, F_j^{(2)})$, which stores the similarity between the i-th point of the first shape and the j-th point of the second shape.

Shape Matching via Linear Assignment

The goal of shape matching is to find corresponding points between two shapes. This is necessary because the feature representation uses a specific parametrization.

One way of formulating this problem is to look for a **permutation** $\pi\colon \{1,\dots,N\} \to \{1,\dots,N\}$ such that

$$E(\pi) = \sum_{i=1}^{N} D_{i,\pi(i)}$$

is minimized.

In other words, we assign to each shape point of the first shape a unique point of the second shape and the cost that we assign to this **assignment** depends **"linearly"** on this choice.
Therefore, this problem is called **Linear Assignment Problem (LAP)**.

Hungarian Method

Shape Matching via Linear Assignment

The LAP has to optimize a function over the space of all permutation. Since there are $N!$ different permuations, it is not clear whether this problem can be solved in polynomial time.

In 1955 Kuhn presented a method that has a time complexity $\mathcal{O}(N^4)$. 1957, Munkres improved the running time to $\mathcal{O}(N^3)$. Kuhn's original work was based on the work of the Hungarians Kőnig and Egerváry. For that reason, the method is sometimes referred as the **Kuhn-Munkres method** or the **Hungarian method**.

The main idea is to change the entries of the non-negative cost matrix D in order to simplify the problem. If there is a permutation π such that $D_{i,\pi(i)} = 0$, we know that we found the global optimum.

An important observation is that by adding a value $a \in \mathbb{R}$ to one row or to one column, we change the value of the minimum by a, but the optimal permutation is still the same.

Trivial Solutions

The following cost matrices are minimized by any permutation. Why?

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

1	1	1	1
0	0	0	0
0	0	0	0
0	0	0	0

1	1	1	1
4	4	4	4
2	2	2	2
3	3	3	3

0	0	3	0
1	1	4	1
0	0	3	0
0	0	3	0

1	1	6	1
4	4	9	4
2	2	7	2
3	3	8	3

1	2	3	4
5	6	7	8
1	2	3	4
5	6	7	8

Example

90	75	75	80	
35	85	55	65	
125	95	90	105	
45	110	95	115	

$\Rightarrow$ 245+

C	C	C		
15	*	0	5	
*	50	20	30	
35	5	*	15	
0	65	50	70	

- For each row r: Find the minimum a_r.
- Subtract from each row r its minimum a_r.
- For each "0" in the matrix, replace it by a * iff there is no * in the same column or row.
- Mark each column that contains a *.
- Iff every column is marked, the stars form an optimal permutation.
- Otherwise, find the minimal entry $a \geqslant 0$ of the non-covered entries.

Example

250+

C		C		
15	*	0	/	C
*	50	20	25	
35	5	*	10	
0	65	50	65	

$\Rightarrow$ 255+

C				
20	*	5	/	C
*	45	20	20	
35	/	*	5	C
0	60	50	60	

- Subtract a from each (unmarked) row and add it to each marked column.
- Replace one zero of the uncovered entries with /. Call its row r.
- If there is a * at position (c, r), unmark the column c and mark row r.
- Find the minimal entry $a \geqslant 0$ of the non-covered entries.

- Subtract a from each unmarked row and add it to each marked column.
- Replace one zero of the uncovered entries with /. Call its row r.
- If there is a * at position (c, r), unmark the column c and mark row r.
- Find the minimal entry $a \geqslant 0$ of the non-covered entries.

Example

275+

40	*	5	/	C
*	25	0	/	C
55	/	*	5	C
/	40	30	40	

$\Rightarrow$ 275+

40	*	5	0
0	25	0	*
55	0	*	5
*	40	30	40

- Subtract a from each (unmarked) row and add it to each marked column.
- Replace one zero of the uncovered entries with /. Call its row r.
- If there is a * at position (c, r), unmark the column c and mark row r.
- *Find the minimal entry $a \geqslant 0$ of the non-covered entries.*

- *Subtract a from each unmarked row and add it to each marked column.*
- Replace one zero of the uncovered entries with /. Call its row r.
- If there is no * in row r, increase the amount of * via back-tracking.
- If the amount of * is maximal, they form the optimal permutation.

Hungarian Method

1. Subtract from each row its minimum. $\Rightarrow D_{i,j} \geqslant 0$.
2. Replace each zero with a * as long as there is no * in that row or column.
3. Mark each *-column. If N columns are marked go to Step 12.
4. Compute the minimum a of the unmarked entries.
5. Subtract a from the unmarked entries and add it to the twice marked entries.
6. Find an unmarked "0" at position (r, c_0) and replace it with /.
7. If there is a * at position (c, r), unmark column c, mark row r and go to Step 4.
8. If there is a * at position (r_0, c_0), there is a / at position (r_1, c_0). This back-tracking terminate with a /.
9. Exchanging the back-tracked / and * increases the amount of * by 1.
10. Unmark all columns and rows and replace every / with a 0.
11. If we have N *, go to Step 12. Otherwise go to Step 4.
12. The N stars in the matrix define the optimal permutation.

Features

- Belongie et al., *Shape Matching and Object Recognition Using Shape Context*, 2002, IEEE TPAMI (24) 24, 509–521.
- Manay et al., *Integral Invariants and Shape Matching*, 2006, IEEE TPAMI (28) 10, 1602–1618.

Hungarian Method

- Kuhn, *The Hungarian Method for the Assignment Problem*, 1955, Naval Research Logistics Quatery 2, 83–97.
- Munkres, *Algorithms for the Assignment and Transportation Problems*, 1957, Journal SIAM (5) 1, 32–38.