Analysis of 3D Shapes (IN2238)

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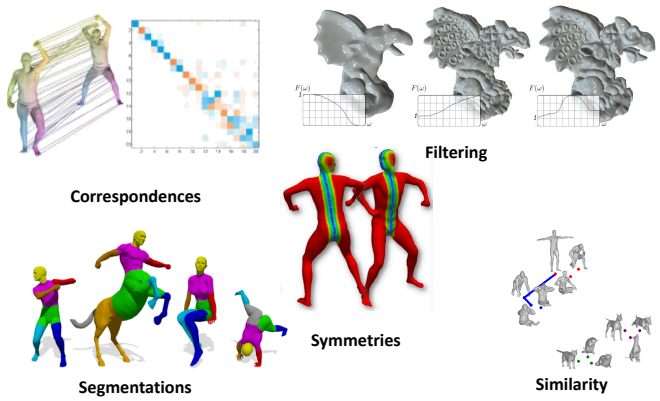
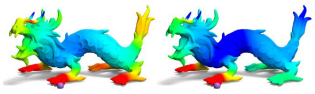
Summer Semester 2016

Outline

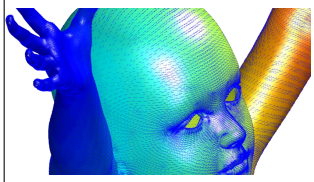


Tools



Pointwise descriptors

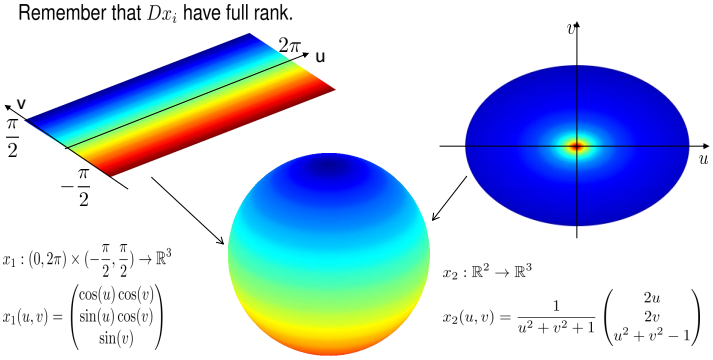
Pairwise descriptors / distances



Differential operators and PDEs

Recap submanifold

In the following weeks we will consider 2-dimensional submanifolds embedded in $\mathbb{R}^3$ given by smooth coordinate mappings $(x_i, U_i \subset \mathbb{R}^2)$.
Remember that Dx_i have full rank.



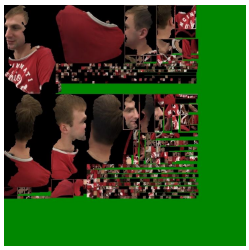
$x_1 : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$

$$x_1(u,v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$

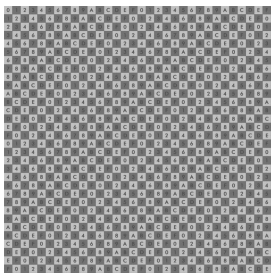
$x_2 : \mathbb{R}^2 \to \mathbb{R}^3$

$$x_2(u,v) = \frac{1}{u^2+v^2+1} \begin{pmatrix} 2u \\ 2v \\ u^2+v^2-1 \end{pmatrix}$$

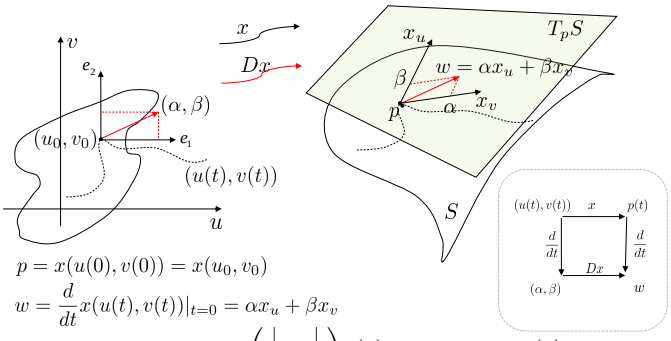
Texture maps



Texture maps



Recap Tangent Space



$$p = x(u(0), v(0)) = x(u_0, v_0)$$

$$w = \frac{d}{dt} x(u(t), v(t))|_{t=0} = \alpha x_u + \beta x_v$$

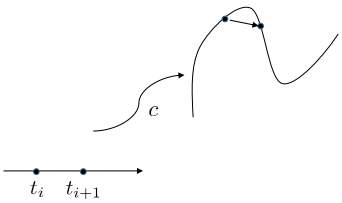
$$= \begin{pmatrix} | & | \\ x_u & x_v \\ | & | \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = Dx(u_0, v_0) \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

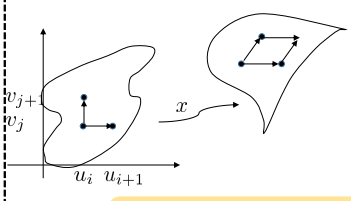
Integrating functions

We want to be able to measure the area of a surface S and integrate scalar functions $f : S \to \mathbb{R}$.

Measuring the area of a surface corresponds to integrating the constant 1-function. We will therefore directly investigate integration of functions.
As in the 1-D (curve) case we transfer the problem of integrating over a manifold to the easier integration in the parameter space.

Surface Integral 1





$$l(t_i) = \|c(t_{i+1}) - c(t_i)\|$$

$$a(u_i, v_j) = area(\begin{pmatrix} x(u_{i+1}, v_j) - x(u_i, v_j) \\ x(u_i, v_{j+1}) - x(u_i, v_j) \end{pmatrix}^T)$$

$$\int_\Gamma f(s)ds = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(c(t_i))l(t_i)$$

$$= \int_I f(c(t)) \cdot \sqrt{\det(c'(t)^\top c'(t))}dt$$

$$\int_S f(p)dp = \lim_{N\to\infty} \sum_{i,j=1}^{N} f(x(u_i, v_j))a(u_i, v_j)$$

$$= \int_U f(x(u,v)) \cdot area\,(Dx)\, dudv$$

Rotating the parallelogram

Our goal is to derive the area of a parallelogram that is spanned by two vectors $\vec{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T$ and $\vec{w} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}^T$. Intuitively the area of a parallelogram should be invariant to rigid motions. We have already used this by placing the parallelogram at the origin.

We can now apply a rotation $R \in SO(3)$ to the parallelogram, such that

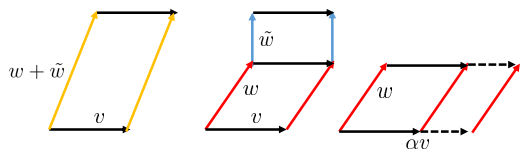
$$R \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix} = \begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \end{pmatrix}$$

is a parallelogram in the x_1x_2-plane having the same area as the original one.

Area of parallelogram in 2D

It turns out to be beneficial to introduce a signed area function $\hat{a} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with the following properties:

- $\hat{a}$ is linear in the first component: $\hat{a}(v + \alpha\tilde{v}, w) = \hat{a}(v, w) + \alpha\hat{a}(\tilde{v}, w)$
- $\hat{a}$ is linear in the second component: $\hat{a}(v, w + \alpha\tilde{w}) = \hat{a}(v, w) + \alpha\hat{a}(v, \tilde{w})$
- $\hat{a}$ is alternating: $\hat{a}(v, v) = 0 \Leftrightarrow \hat{a}(v, w) = -\hat{a}(w, v)$
- $\hat{a}$ is normalized: $\hat{a}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = 1$



Area of parallelogram in 2D

The area of the parallelogram spanned by $\tilde{v}$ and $\tilde{w}$ will then be given by $area(\tilde{v}, \tilde{w}) = |\hat{a}(\tilde{v}, \tilde{w})|$.

$$\begin{aligned} area(\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}, \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}) &= |\hat{a}(\tilde{v}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{v}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tilde{w}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{w}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix})| \\ &= \tilde{v}_1\tilde{w}_1\hat{a}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + \tilde{v}_1\tilde{w}_2\hat{a}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &+ \tilde{v}_2\tilde{w}_1\hat{a}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + \tilde{v}_2\tilde{w}_2\hat{a}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &= |\tilde{v}_1\tilde{w}_2 - \tilde{v}_2\tilde{w}_1| = |\det \begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \end{pmatrix}| \end{aligned}$$

Area of parallelogram in 3D

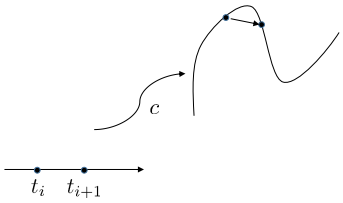
Next we consider the original parallelogram centered at the origin and spanned by $v = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T$ and $v = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}^T$.

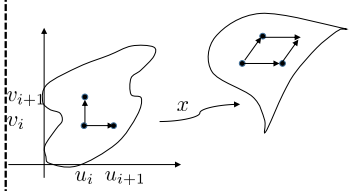
$$area(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}) = area(R \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, R \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix})$$

$$= area(\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ 0 \end{pmatrix}) = |\det(\begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \end{pmatrix})| = \sqrt{\det(\begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \end{pmatrix}^T \begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \end{pmatrix})}$$

$$= \sqrt{\det(\begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} \tilde{v}_1 & \tilde{w}_1 \\ \tilde{v}_2 & \tilde{w}_2 \\ 0 & 0 \end{pmatrix})} = \sqrt{\det(\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix}^T \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix})}$$

Surface Integral 2





$$l(t_i) = \|c(t_{i+1}) - c(t_i)\|$$

$$a(u_i, v_j) = area(\begin{pmatrix} x(u_{i+1}, v_i) - x(u_i, v_i) \\ x(u_i, v_{i+1}) - x(u_i, v_i) \end{pmatrix}^T)$$

$$\int_\Gamma f(s)ds = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(c(t_i))l(t_i)$$

$$= \int_I f(c(t)) \cdot \sqrt{\det(c'(t)^\top c'(t))}dt$$

$$\int_M f(p)dp = \lim_{N\to\infty} \sum_{i,j=1}^{N} f(x(u_i, v_j))a(u_i, v_j)$$

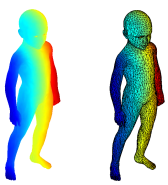
$$= \int_U f(x(u,v)) \cdot \sqrt{\det((Dx)^T (Dx))}dudv$$

Discrete surfaces

As a matter of fact, we do not really work with manifolds but with discrete representations of manifolds (mostly triangular meshes).

However, we will still be able to define meaningful quantities which approximate well (in some sense) their continuous counterparts.

Discretizing the notions of differential geometry to work with meshes is the main task of discrete differential geometry.

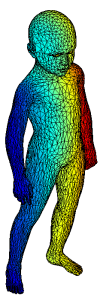


Discrete surfaces

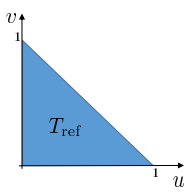
We will in the following treat each triangle of a mesh as an affine surface patch.

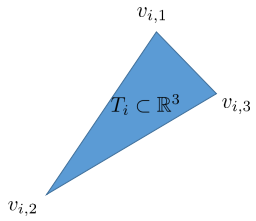
The coordinate maps $(U_i, x_i)_{i=1,\dots,m}$ are given by

$$U_i = T_{\text{ref}} = \{(u,v) \in (0,1)^2 | u + v < 1\}$$
$$x_i(u,v) = v_{i,1} + u \cdot (v_{i,2} - v_{i,1}) + v \cdot (v_{i,3} - v_{i,1})$$



n vertices
m triangles

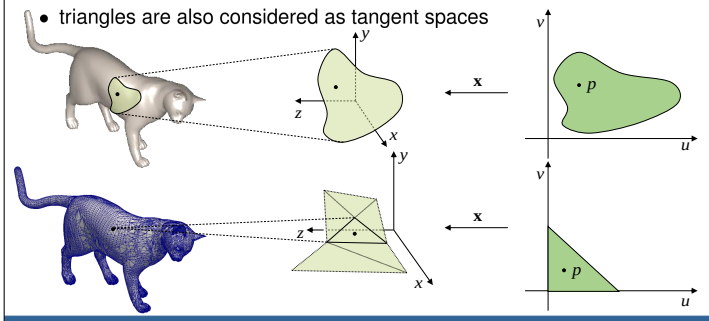




Discrete surfaces

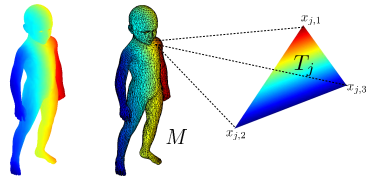
Notice that we are *cheating* a bit:

- strictly speaking, edges are not covered
- surface is not *smooth*
- triangles are also considered as tangent spaces



Discretization: functions

Not only the considered surfaces but also functions defined on them have to be dicretized. The simplest approach is to store the function values $\mathbf{f}_i = f(v_i)$ at the vertices in a vector $\mathbf{f} \in \mathbb{R}^n$.



Within the triangles we assume the function f to act linearly. The function is defined on the complete mesh and not only at the vertices.

Notice the difference between the vector $\mathbf{f}$ and the function f.

This is by far not the only way to discretize surfaces and functions defined on them but will accompany us for the next weeks.

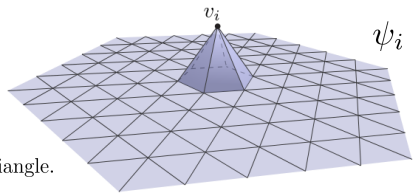
Hat functions

The entries of the vector $\mathbf{f}$ have a second interpretation as coefficients of the function f in the *hat basis* $\{\psi_i | i = 1 \dots n\}$.

$$f(x) = \sum_i \mathbf{f}_i \psi_i(x)$$

$$\psi_i(v_j) = \delta_{ij}$$

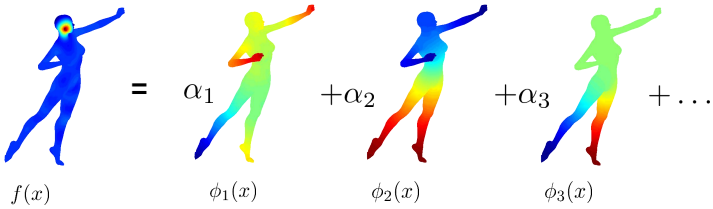
ψ_i acts linear inside each triangle.



We call the space $PL(S) := span(\psi_i)$ the space of piecewise linear functions on the mesh S.

Different basis

We will soon equip the space $PL(S)$ with a different basis.



Notice that the coefficients α_i in this basis do not correspond to function values anymore.

Integrating a PL function

We will now bring the two topics of todays lecture together and integrate functions $f \in PL(S)$.

Let $S = \cup_{i=1}^m T_i$ be a triangular mesh with m triangles and n vertices. Then

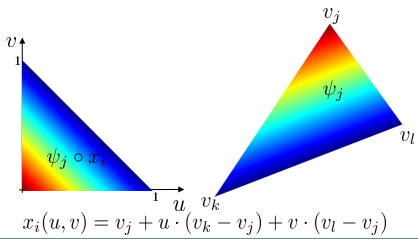
$$\int_S f(p)dp = \sum_{i=1}^m \int_{T_i} f(p)dp = \sum_{i=1}^m \int_{T_i} \sum_{j=1}^n \mathbf{f}_j \psi_j(p)dp$$

$$= \sum_{j=1}^n \mathbf{f}_j \underbrace{\sum_{i=1}^m \int_{T_i} \psi_j(p)dp}_{\mathbf{a}_j}$$

$$= \mathbf{a}^T \mathbf{f}$$

$\mathbf{a}_j = ?$

Homework



$$x_i(u,v) = v_j + u \cdot (v_k - v_j) + v \cdot (v_l - v_j)$$

Inner product (smooth)

With the possibility of integration by hand we can define the inner product between functions $f, g : S \to \mathbb{R}$ defined on a manifold

$$\langle f, g \rangle_S := \int_S f(p)g(p)dp = \int_U f(x(u,v))g(x(u,v))\sqrt{\det(Dx^T Dx)}dudv$$

The alternative basis functions will turn out to be orthonormal with respect to this inner product.

Inner product (discrete)

In the discrete setup, where we are given two functions $f, g \in PL(S)$ by their coefficientvectors (in the hat basis) $\mathbf{f}$ and $\mathbf{g}$, we can again make use of the linearity of the integral:

$$\langle f, g \rangle_S = \int_S \left(\sum_j \mathbf{f}_j \psi_j(p) \right) \left(\sum_k \mathbf{g}_k \psi_k(p) \right) dp$$

$$= \sum_j \sum_k \mathbf{f}_j \left(\int_M \psi_j(p)\psi_k(p)dp \right) \mathbf{g}_k$$

$$= \sum_j \sum_k \mathbf{f}_j \mathbf{M}_{jk} \mathbf{g}_k = \mathbf{f}^T \mathbf{M} \mathbf{g}$$

$$\mathbf{M}_{jk} = \int_S \psi_j(p)\psi_k(p)dp = ?$$

Homework