Analysis of 3D Shapes (IN2238)

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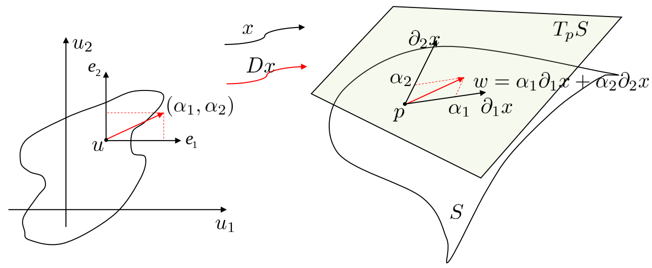
Recap integration

Last week we have seen how to integrate scalar functions $f : S \to \mathbb{R}$ defined on a surface:

$$\int_S f(p)dp = \int_U f(x(u)) \cdot \sqrt{\det((Dx)^T (Dx))}du = \int_U f(x(u)) \cdot \sqrt{\det g}du$$

The matrix $g = Dx^T Dx$ is called **first fundamental form** of x.



First fundamental form

Each coordinate map $x : U \to \mathbb{R}$ comes with its own first fundamental form $g = Dx^T Dx$.

Notice that g is in general not constant (as is Dx) but is a (smooth) function $g : U \to \mathbb{R}^{2\times 2}$.

$$g(u) = \begin{pmatrix} g_{11}(u) & g_{12}(u) \\ g_{21}(u) & g_{22}(u) \end{pmatrix} = \begin{pmatrix} \langle \partial_1 x(u), \partial_1 x(u)\rangle & \langle \partial_1 x(u), \partial_2 x(u)\rangle \\ \langle \partial_1 x(u), \partial_2 x(u)\rangle & \langle \partial_2 x(u), \partial_2 x(u)\rangle \end{pmatrix}$$

Since Dx has full rank the first fundamental form is a symmetric positive definit.

Sometimes g is also called **Riemannian metric**.

Notation:

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$

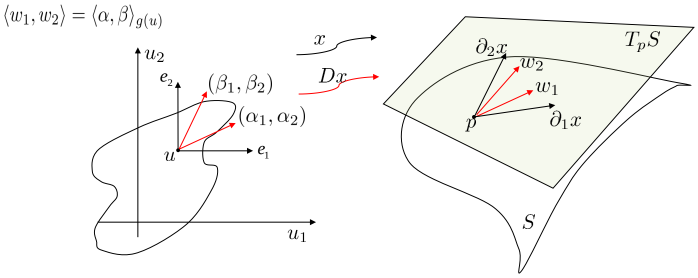
Measuring length and angles

The first fundamental form g defines an inner product such that we can express length and angles of tangent vectors in local coordinates.

Let $w_1 = \alpha_1\partial_1 x(u) + \alpha_2\partial_2 x(u)$ and $w_2 = \beta_1\partial_1 x(u) + \beta_2\partial_2 x(u)$ be two tangent vectors at $x(u) = p \in S$.

$$\|w_1\|^2 = \langle Dx \cdot \alpha, Dx \cdot \alpha\rangle_{\mathbb{R}^3} = \alpha^T g(u)\alpha = \langle \alpha, \alpha\rangle_{g(u)}$$

$$\langle w_1, w_2\rangle = \langle \alpha, \beta\rangle_{g(u)}$$



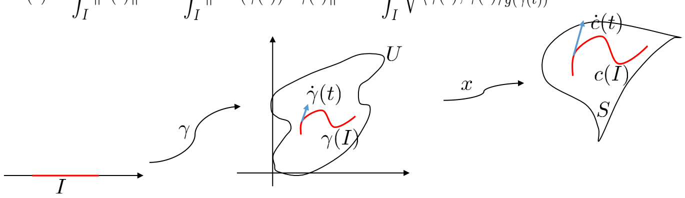
Length of a curve

The first fundamental form gives us the posiibility to measure the length of a curve defined on the manifold S.

Let I be an interval and $\gamma : I \to U$ be a curve in the parameter domain with $\|\dot\gamma\| \neq 0$. Then $c = x \circ \gamma : I \to S$ defines a curve on the manifold, with $\|\dot c\| \neq 0$ (*Why?*).

The length of c is given by

$$L(c) = \int_I \|\dot c(t)\|\, dt = \int_I \|Dx(\gamma(t)) \cdot \dot\gamma(t)\|\, dt = \int_I \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{g(\gamma(t))}}dt$$



Angle between curves

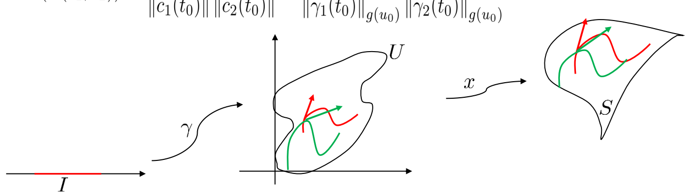
Let $I = (a, b)$ be an interval and $\gamma_1, \gamma_2 : I \to U$ be two curves in the parameter domain intersecting at $u_0 = \gamma_1(t_0) = \gamma_2(t_0)$.

We can now measure the angles between γ_1 and γ_2 at u_0:

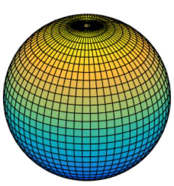
$$\cos(\angle(\gamma_1, \gamma_2)) = \frac{\langle \dot\gamma_1(t_0), \dot\gamma_2(t_0)\rangle}{\|\dot\gamma_1(t_0)\|\,\|\dot\gamma_2(t_0)\|}$$

For the angles between $c_1 = x \circ \gamma_1$ and $c_2 = x \circ \gamma_2$ at $p = x(q)$ we observe:

$$\cos(\angle(c_1, c_2)) = \frac{\langle \dot c_1(t_0), \dot c_2(t_0)\rangle}{\|\dot c_1(t_0)\|\,\|\dot c_2(t_0)\|} = \frac{\langle \dot\gamma_1(t_0), \dot\gamma_2(t_0)\rangle_{g(u_0)}}{\|\dot\gamma_1(t_0)\|_{g(u_0)}\,\|\dot\gamma_2(t_0)\|_{g(u_0)}}$$



Example: Sphere



$$x(u) = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)^T$$

$$U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$Dx = \begin{pmatrix} -\sin u_1 \cos u_2 & -\cos u_1 \sin u_2 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ 0 & \cos u_2 \end{pmatrix}$$

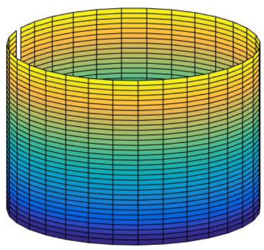
$$g = \begin{pmatrix} \cos^2 u_2 & 0 \\ 0 & 1 \end{pmatrix}$$

If the off-diagonal entries of the first fundamental form vanish, we call x an orthogonal parametrization.

Curves that intersect in a right angle in U also intersect in a right angle on the surface.

This in particular applies to the parameter lines.

Example: Cylinder



$$x(u, v) = (\cos u, \sin u, v)^T$$

$$U = (0, 2\pi) \times \mathbb{R}$$

$$Dx = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If the first fundamental form is the identity matrix, we call x an isometric parametrization.

Angles between curves and length of curves in the parameter domain U are preserved under x.

Isometric parametrization

If x is an isometric parametrization, i.e. $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we see:

$$L(c) = \int_I \|\dot{c}(t)\|\,dt = \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_{g(\gamma(t))}}dt = \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle}dt = L(\gamma)$$

This means: isometric parametrizations preserve length.

It is easy to see that isometric parametrizations also preserve angles and areas $(\det(g) = 1)$.

Homework

There are angle-preserving parametrizations that do not preserve areas.

There are area-preserving maps that do not preserve angles.

Geodesic distance

The geodesic distance $d_S(x, y)$ measures the length of the shortest path on the surface S connecting $x \in S$ and $y \in S$.

$d_S(x, y) = \inf\{L(c) | c : (a, b) \to S,\ c(a) = x,\ c(b) = y\}$

$d_S : S \times S \to \mathbb{R}^+$ is a metric.

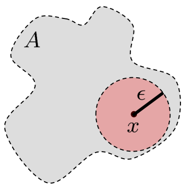
Open sets

Let (S, d_S) be a metric space. We define the open ball of radius r around $x \in S$ as

$B_r(x) := \{y \in S | d_S(x, y) < r\}$

A subset $A \subset X$ is called open if for any $x \in A$ there exists a $\epsilon < 0$ such that $B_\epsilon(x) \subset A$.

In particular S itself is open.



Compactness

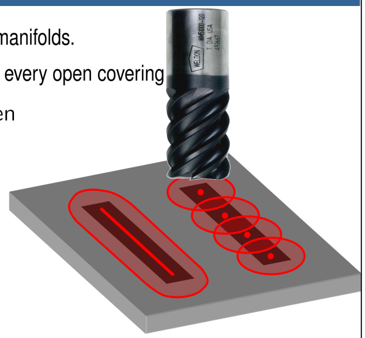
The shapes we consider will be compact manifolds.

A metric space (S, d_S) is called compact if every open covering

$$\bigcup_\alpha U_\alpha = S \qquad U_\alpha \subset S \text{ open}$$

has a finite subcovering:

$$\bigcup_{i=1}^{N} U_{\alpha_i} = S$$



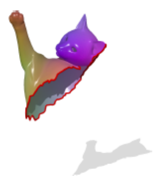
More intuitiv:

- closed: every Cauchy sequence $(x_n) \subset S$ has a limit $x \in S$
- bounded: $\operatorname{diam}(S) = \sup_{x,y \in S} d_S(x, y)$ is finite

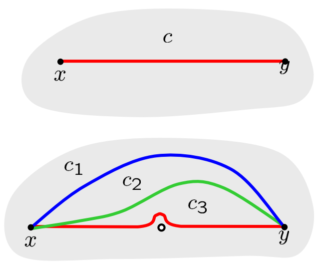
Consequences

The shapes we consider are at the same time open and compact.

Some consequences:



no boundary



complete

$d_S(x, y) = \min_c L(c)$

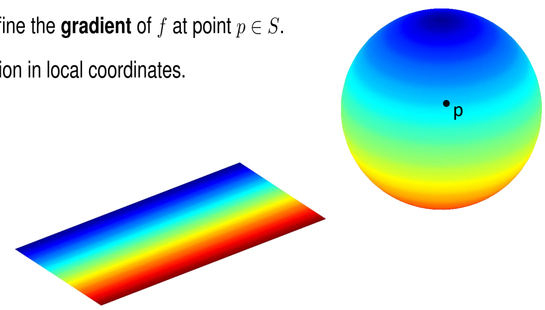
Gradient of a function

Consider a surface S with parametrization (x, U).

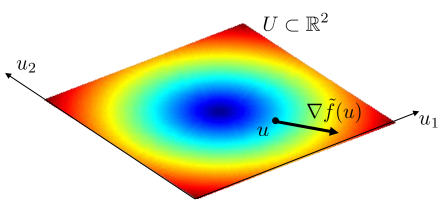
A function $f : S \to \mathbb{R}$ is called differentiable if $\tilde{f} = f \circ x : U \to \mathbb{R}$ is differentiable.

We want to define the **gradient** of f at point $p \in S$.

Goal: Expression in local coordinates.



Gradient in euclidean space



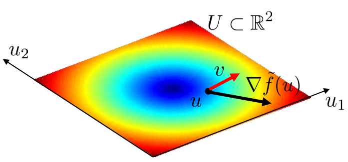
The gradient of a differentiable function $\tilde{f} : U \to \mathbb{R}$ is the vectorfield

$$\nabla \tilde{f}(u) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u_1}(u) \\ \frac{\partial \tilde{f}}{\partial u_2}(u) \end{pmatrix}$$

Geometric meaning

Geometric meaning of the gradient:

- the vector that points in the **direction of steepest increase** of $\tilde{f}$
- its length measures the strength of increase
- relationship with the differential of $\tilde{f}$:



$$d\tilde{f}(u)(\vec{v}) = \lim_{h \to 0} \frac{\tilde{f}(p + h\vec{v}) - \tilde{f}(p)}{h}$$

$$= \frac{d}{dh} \tilde{f}(p + h\vec{v})|_{h=0}$$

$$= \langle \nabla \tilde{f}(u), \vec{v} \rangle$$

directional derivative of $\tilde{f}$ at u, along direction v

Riesz Representation

Dual space

Let X be a vectorspace. Then we denote by

$$X^* = \{\phi : X \to \mathbb{R} | \phi \text{ linear}\}$$

the **dual space** of X.

Riesz Representation theorem

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space (complete, inner product).
Then for each *continuous* $\phi \in H^*$ there exists a unique $y \in H$ such that

$$\phi(x) = \langle y, x\rangle \quad \forall x \in H$$

Differential of f

The differential of a function $f : S \to \mathbb{R}$ at a point $p \in S$ is the linear mapping $df(p) : T_pM \to \mathbb{R}$ satisfying

$$df(p)[v] = \lim_{t\to 0} \frac{f(c(t)) - f(p)}{t}$$

for all curves $c : (-\varepsilon, \varepsilon) \to S$ with $c(0) = p$ and $\dot{c}(0) = v$.

Uniqueness and Linearity

$c_1(0) = c_2(0) = p$

$\dot{c}_1(0) = \dot{c}_2(0) = v$

By defining the preimages $\gamma_i(t) = x^{-1} \circ c_i(t)$ and as usual $\tilde{f} = f \circ x$ we get

$$df(p)[v] = \lim_{t\to 0} \frac{f(c_i(t)) - f(p)}{t} = \lim_{t\to 0} \frac{\tilde{f}(\gamma_i(t)) - \tilde{f}(u)}{t} = \frac{d}{dt}\tilde{f}(\gamma_i(t))|_{t=0}$$

$$= \langle \nabla\tilde{f}(0), \dot{\gamma}_i(0)\rangle = \langle \nabla\tilde{f}(0), (Dx)^{-1}\dot{c}_i(0)\rangle = \boxed{\langle \nabla\tilde{f}(u), (Dx)^{-1}} v\rangle$$

Gradient

Definition

Let $f : S \to \mathbb{R}$ be a differentiable function. The gradient $\nabla f(p)$ at $p \in S$ is the unique element of T_pS such that

$$\langle \nabla f(p), v\rangle = df(p)[v]$$

(Possible due to Riesz representation theorem)

The gradient in local coordinates

Given $\nabla\tilde{f}$ and g, the coefficients α of $\nabla f = Dx \cdot \alpha \in T_pS$ are given by $\alpha = g^{-1}\nabla\tilde{f}(x^{-1}(p))$

Let β be the coefficients of $v \in T_pS$. Then

$$df(p)[v] = \langle \nabla\tilde{f}(u), \beta\rangle \;\; = \langle \nabla f, v\rangle = \langle \alpha, \beta\rangle_{g(u)}$$