Analysis of 3D Shapes (IN2238)

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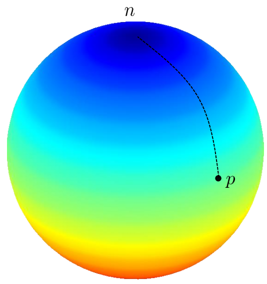
Example

Consider the function $f : \mathbb{S}^2 \setminus \{n\} \to \mathbb{R}$ that assigns each point p on the unit sphere its distance to the northpole n:

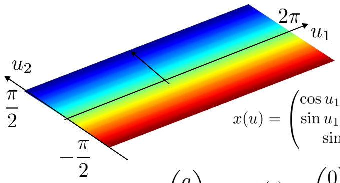
$$f(p) = d_{\mathbb{S}^2}(n, p)$$



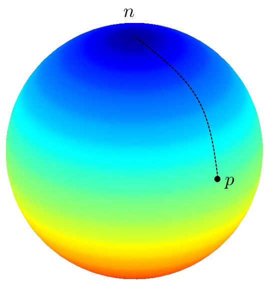
Without proof:

The distance is realized by the path sketched on the right.

Distance



$$x(u) = \begin{pmatrix} \cos u_1 \cos u_2 \\ \sin u_1 \cos u_2 \\ \sin u_2 \end{pmatrix}$$



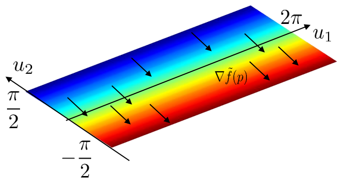
$$\gamma(t) = \begin{pmatrix} a \\ t \end{pmatrix} \qquad \dot{\gamma}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$t \in (\arcsin(p_3), \frac{\pi}{2})$$

$$L(c) = \int_{\arcsin p_3}^{\frac{\pi}{2}} \sqrt{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2(-t) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} dt = \frac{\pi}{2} - \arcsin p_3$$

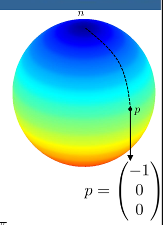
We have not shown that this is indeed the shortest path connecting n and p.

Gradients distance



$$u = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

$$\nabla f(p) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$



$$p = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$x(u) = \begin{pmatrix} \cos u_1 \cos u_2 \\ \sin u_1 \cos u_2 \\ \sin u_2 \end{pmatrix}$$

$$f(p) = -\arcsin(x_3) + \frac{\pi}{2}$$

$$\tilde{f}(u) = -\arcsin(\sin u_2) + \frac{\pi}{2} = -u_2 + \frac{\pi}{2}$$

$$Dx = \begin{pmatrix} -\sin u_1 \cos u_2 & -\cos u_1 \sin u_2 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ 0 & \cos u_2 \end{pmatrix}$$

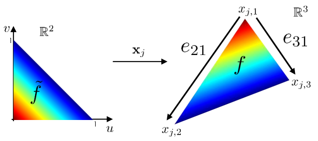
$$\nabla \tilde{f}(u) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$g(u) = \begin{pmatrix} \cos^2 u_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nabla f(p) = Dx \cdot g^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \cos u_1 \sin u_2 \\ \sin u_1 \sin u_2 \\ -\cos u_2 \end{pmatrix}$$

Discretization: gradient

$$\mathbf{x}_j(u) = x_{j,1} + u_1(x_{j,2} - x_{j,1}) + u_2(x_{j,3} - x_{j,1})$$



$$\mathrm{Dx} = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

$$\det g_j = 2|T_j|$$

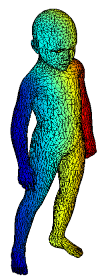
Recall that we model functions on triangular meshes to act linearly within each traingle T_j.
The gradients are thus constant inside each triangle and given (in local coordinates) by

$$\nabla f|_{T_j} = Dx \cdot g^{-1} \nabla \tilde{f} = Dx \cdot \frac{1}{2|T_j|} \begin{pmatrix} \|e_{31}\|^2 & -\langle e_{21}, e_{31} \rangle \\ -\langle e_{21}, e_{31} \rangle & \|e_{21}\|^2 \end{pmatrix} \begin{pmatrix} f(x_{j,2}) - f(x_{j,1}) \\ f(x_{j,3}) - f(x_{j,1}) \end{pmatrix}$$

Weak derivatives

Strictly speaking piecewise linear functions are not differentiable (in the classical sense).

However it is possible to define **weak derivatives** that generalize classical derivatives.



Main information:

If a function is continuous and piecewise (classically) differentiable (up to sets of measure 0), classical and weak derivatives coinside.

Closure and support

Closure of a set

Let $A \subset \mathbb{R}^n$. The closure of A is the (closed) set

$$\overline{A} = \{x \in \mathbb{R}^n | \exists (x_n)_{n \in \mathbb{N}} \subset A \text{ with } x_n \to x\}$$

Support of a function

Let $S \subset \mathbb{R}^n$ be open. The support of a function $f : S \to \mathbb{R}$ is the set

$$\operatorname{supp} f = \overline{\{x \in S | f(x) \neq 0\}}$$

If $\operatorname{supp} f$ is compact and $\operatorname{supp} f \subset S$ we say that f has compact support in S and write $\operatorname{supp} f \subset\subset S$.

Test function

Test functions

Let again $S \subset \mathbb{R}^n$ be open. The set of testfunctions on S is defined as

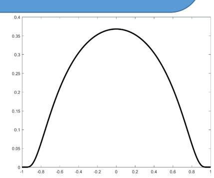
$$C_c^\infty(S) = \{f \in C^\infty(S) | \operatorname{supp} f \subset\subset S\}$$

If S is at the same time compact (as are the shapes we consider), then $C_c^\infty(S) = C^\infty(S)$

Example

Consider the function

$$\phi(x) = \begin{cases} \exp(-\frac{1}{1-x^2}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



Clearly $\operatorname{supp} \phi = [-1, 1]$ is compact.

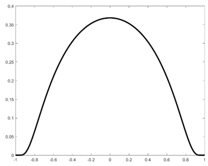
Since $[-1, 1]$ is not contained in the open interval $(-1, 1)$, $\phi \notin C_c^\infty((-1, 1))$ but $\phi \in C_c^\infty(\mathbb{R})$ and $\phi \in C_c^\infty((a, b)) \forall [-1, 1] \subset (a, b)$.

Properties

If the open set $S \subset \mathbb{R}^n$ has a boundary and $f \in C_c^\infty(S)$, then there is an $\varepsilon > 0$ such that f vanishes for all points that are closer then ε to the boundary ∂S:

$$\operatorname{dist}(\operatorname{supp} f, \partial S) = \varepsilon > 0$$

In the previous example $\partial S = \{a, b\}$ and $\varepsilon = \min\{-(a-1), b-1\}$



As a consequence f and all its derivatives vanish at the boundary of S.

Weak derivative 1D

Weak derivative

A function $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$ is called weakly differentiable if there exists a function $g : (a,b) \to \mathbb{R}$ such that

$$\int_a^b f(x)\phi'(x)dx = -\int_a^b g(x)\phi(x)dx$$

for all testfunctions $\phi \in C_c^\infty((a,b))$. g is then called the weak derivative of f.

The weak derivative is unique (up to null sets).

If $f \in C^1$ weak and classical derivative coincide:

$$\int_a^b f(x)\phi'(x)dx = f(x)\phi(x)|_a^b - \int_a^b f'(x)\phi(x)dx = -\int_a^b f'(x)\phi(x)dx$$

Example

Consider the continuous, piecewise differentiable function $f : (-1,1) \to \mathbb{R}$

$$f(x) = |x|$$

Let $\phi \in C_c^\infty(-1,1)$ be a test function. Then

$$\int_{-1}^1 f(x)\phi'(x)dx = -\int_{-1}^0 x\phi'(x)dx + \int_0^1 x\phi'(x)dx$$

$$= \left(-x\phi(x)|_{-1}^0 + \int_{-1}^0 \phi(x)dx\right) + \left(x\phi(x)|_0^1 - \int_0^1 \phi(x)dx\right)$$

$$= -\int_{-1}^1 g(x)\phi(x)dx$$

$$g(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$$

A Lebesgue space

Square integrable functions

Let S be a compact manifold. Then we denote by

$$L^2(S) = \{f : S \to \mathbb{R} | \int_S |f(p)|^2 dp < \infty\}$$

the space of square integrable functions on S.

Let $f, g \in L^2(S)$. Then

$$\langle f, g\rangle_{L^2(S)} = \int_S f(p)g(p)dp$$

defines an inner product.

Another inner product

A Sobolev space

Let S be a compact manifold and denote by ∇f the weak derivative of f. Then on

$$H^1(S) = \{f : S \to \mathbb{R} | \int_S |f(p)|^2 dp < \infty, \int_S |\nabla f(p)|^2 dp < \infty\}$$

we can define the inner product

$$\langle f, g\rangle_{H^1(S)} = \int_S f(p)g(p)dp + \int_S \langle \nabla f(p), \nabla g(p)\rangle dp$$

In general a Sobolev space is denoted by $W^{k,p}(S)$ and consists of functions that are k times weakly differentiable with all derivatives having finite L^p norm.

For the cases $p = 2$ one writes $W^{k,p} = H^k$ to denote that these are Hilbert spaces.

The cotangent matrix

The space $PL(S) = \operatorname{span}\{\psi_i | i = 1 \dots n\}$ of piecewise linear functions on a triangular mesh $S = (\{v_i\}_{i=1}^n, \{T_j\}_{j=1}^m)$ is a subset of $H^1(S)$.

Let $f = \sum \mathbf{f}_i \psi_i$ and $g = \sum \mathbf{g}_j \psi_j$. Then their H^1 inner product is given by

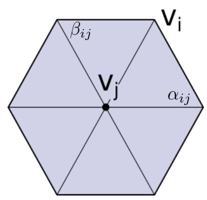
$$\langle f, g\rangle_{H^1(S)} = \langle f, g\rangle_{L^2(S)} + \int_S \langle \nabla f, \nabla g\rangle dp = \mathbf{f}^T\mathbf{M}\mathbf{g} + \mathbf{f}^T\mathbf{C}\mathbf{g}$$

The entries of $\mathbf{C} \in \mathbb{R}^{n \times n}$ are given by:

$$C_{ij} = \int_S \langle \nabla \psi_i, \nabla \psi_j\rangle dp$$

In the **Homework** you show that

$$C_{ij} = \begin{cases} -\frac{1}{2}(\cot \alpha_{ij} + \beta_{ij}) & \text{if } v_i \text{ and } v_j \text{ share an edge} \\ 0 & \text{else} \end{cases}$$



Divergence euclidean

Smooth vector field

A smooth vectorfield on an open $U \subset \mathbb{R}^n$ is a function

$$\alpha : U \to \mathbb{R}^n$$

$$\alpha(u) = \begin{pmatrix} \alpha_1(u) & \dots & \alpha_n(u) \end{pmatrix}^T$$

where the coefficient functions $\alpha_i : U \to \mathbb{R}$ are smooth.

Divergence

The divergence of a vectorfield $\alpha : U \to \mathbb{R}^n$ is defined via:

$$\operatorname{div} \alpha : U \to \mathbb{R}$$

$$\operatorname{div} \alpha(u) = \sum_{i=1}^n \partial_i \alpha_i(u)$$

Adjointness to gradient

Let $\alpha : U \to \mathbb{R}^2$ be a smooth vectorfield on $U = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ and $\tilde{f} \in C_c^\infty(U)$ a test function.

$$\langle \nabla \tilde{f}, \alpha\rangle = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \alpha_1(u)\partial_1\tilde{f}(u) + \alpha_2(u)\partial_2\tilde{f}(u) du_2 du_1$$

$$= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \alpha_1(u)\partial_1\tilde{f}(u) du_1 du_2 + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \alpha_2(u)\partial_2\tilde{f}(u) du_2 du_1$$

$$\stackrel{*}{=} -\int_{a_2}^{b_2}\int_{a_1}^{b_1} \partial_1\alpha_1(u)\tilde{f}(u) du_1 du_2 - \int_{a_2}^{b_2}\int_{a_1}^{b_1} \partial_2\alpha_2(u)\tilde{f}(u) du_1 du_2$$

$$= -\int_U \tilde{f}(u) \operatorname{div} \alpha(u) du = -\langle \tilde{f}, \operatorname{div} \alpha\rangle$$

The same holds for an arbitrary open subset $U \subset \mathbb{R}^2$ and higher dimensions.

$$* \int_{a_2}^{b_2}\int_{a_1}^{b_1} \alpha_1(u)\partial_1\tilde{f}(u) du_1 du_2 = \int_{a_2}^{b_2}\left(\alpha_1(u)\tilde{f}(u)|_{a_1}^{b_1} - \int_{a_1}^{b_1} \partial_1\alpha_1(u)\tilde{f}(u) du_1\right) du_2$$

Smooth vector field

A smooth vectorfield on a compact manifold S is a function

$$V(p) = Dx \cdot \begin{pmatrix} \alpha_1(x^{-1}(p)) \\ \alpha_2(x^{-1}(p)) \end{pmatrix}$$

where the coefficient functions $\alpha_i : U \to \mathbb{R}$ are smooth.

Divergence

The divergence of a smooth vectorfield V is the scalar function $\operatorname{div} V : S \to \mathbb{R}$ defined via

$$\int_S \langle \nabla f, V \rangle dp = -\int_S f(p) \operatorname{div} V(p) dp$$

for all test functions $f \in C_c^\infty(S)$.