Analysis of 3D Shapes (IN2238)

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Closure and support

Closure of a set

Let $A \subset \mathbb{R}^n$. The closure of A is the (closed) set

$$\overline{A} = \{x \in \mathbb{R}^n | \exists (x_n)_{n\in\mathbb{N}} \subset A \text{ with } x_n \to x\}$$

Support of a function

Let $S \subset \mathbb{R}^n$ be open. The support of a function $f : S \to \mathbb{R}$ is the set

$$\operatorname{supp} f = \overline{\{x \in S | f(x) \neq 0\}}$$

If $\operatorname{supp} f$ is compact and $\operatorname{supp} f \subset S$ we say that f has compact support in S and write $\operatorname{supp} f \subset\subset S$.

Test function

Test functions

Let again $S \subset \mathbb{R}^n$ be open. The set of testfunctions on S is defined as

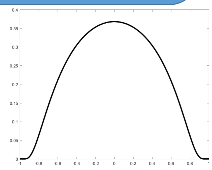
$$C_c^\infty(S) = \{f \in C^\infty(S) | \operatorname{supp} f \subset\subset S\}$$

If S is at the same time compact as are the shapes we consider, then $C_c^\infty(S) = C^\infty(S)$

Example

Consider the function

$$\phi(x) = \begin{cases} \exp(-\frac{1}{1-x^2}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



Clearly $\operatorname{supp} \phi = [-1, 1]$ is compact.

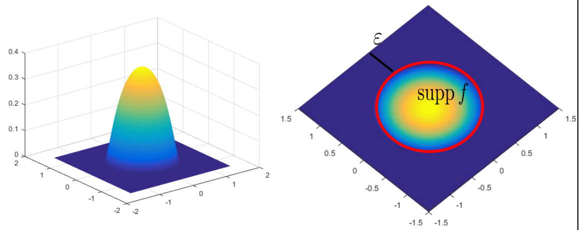
Since $[-1, 1]$ is not contained in the open interval $(-1, 1)$, $\phi \notin C_c^\infty((-1, 1))$ but $\phi \in C_c^\infty(\mathbb{R})$ and $\phi \in C_c^\infty((a, b)) \forall [-1, 1] \subset (a, b)$.

Properties

If the open set $S \subset \mathbb{R}^n$ is bounded and $f \in C_c^\infty(S)$, then there is an $\varepsilon > 0$ such that f vanishes for all points that are closer then ε to the boundary ∂S:

$$\operatorname{dist}(\operatorname{supp} f, \partial S) = \varepsilon > 0$$

As a consequence f and all its derivatives vanish at the boundary of S.



Weak derivative 1D

Weak derivative

A function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is called weakly differentiable if there exists a function $g : (a, b) \to \mathbb{R}$ such that

$$\int_a^b f(x)\phi'(x)dx = -\int_a^b g(x)\phi(x)dx$$

for all testfunctions $\phi \in C_c^\infty((a, b))$. g is then called the weak derivative of f.

The weak derivative is unique (up to null sets).

If $f \in C^1$ weak and classical derivative coincide:

$$\int_a^b f(x)\phi'(x)dx = f(x)\phi(x)|_a^b - \int_a^b f'(x)\phi(x)dx = -\int_a^b f'(x)\phi(x)dx$$

Example

Consider the continuous, piecewise differentiable function $f : (-1, 1) \to \mathbb{R}$

$$f(x) = |x|$$

Let $\phi \in C_c^\infty(-1, 1)$ be a test function. Then

$$\int_{-1}^1 f(x)\phi'(x)dx = -\int_{-1}^0 x\phi'(x)dx + \int_0^1 x\phi'(x)dx$$

$$= \left(-x\phi(x)|_{-1}^0 + \int_{-1}^0 \phi(x)dx\right) + \left(x\phi(x)|_0^1 - \int_0^1 \phi(x)dx\right)$$

$$= -\int_{-1}^1 g(x)\phi(x)dx$$

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Divergence euclidean

Smooth vector field

A smooth vectorfield on an open $U \subset \mathbb{R}^n$ is a function

$$\alpha : U \to \mathbb{R}^n$$

$$\alpha(u) = \begin{pmatrix} \alpha_1(u) & \dots & \alpha_n(u) \end{pmatrix}^T$$

where the coefficient functions $\alpha_i : U \to \mathbb{R}$ are smooth.

Divergence

The divergence of a vectorfield $\alpha : U \to \mathbb{R}^n$ is defined via:

$$\operatorname{div} \alpha : U \to \mathbb{R}$$

$$\operatorname{div} \alpha(u) = \sum_{i=1}^n \partial_i \alpha_i(u)$$

Integration by parts

Fundamental theorem of calculus

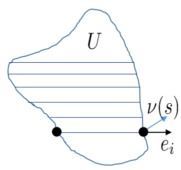
Let $U \subset \mathbb{R}^n$ be open with piecwise smooth ∂U and $f \in C^1(\bar{U})$. Then

$$\int_U \partial_i f(u)du = \int_{\partial U} f(s)\langle e_i, \nu(s)\rangle ds$$

where $\nu : \partial U \to \mathbb{R}^n$ depicts the unit outward normal.

For a vectorfield $\alpha : U \to \mathbb{R}^n$ this implies

$$\int_U \operatorname{div} \alpha(u)du = \int_{\partial U} \langle \alpha(s), \nu(s)\rangle ds$$



The product rule yields

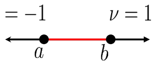
$$\int_U (\partial_i f(u))g(u)du + \int_U f(u)(\partial_i g(u))du = \int_U \partial_i (f(u)g(u))du$$

$$= \int_{\partial U} f(s)g(s)\langle e_i, \nu(s)\rangle ds$$

Examples

Example 1

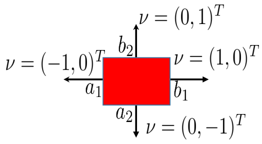
Let $U = (a,b) \subset \mathbb{R}$ with boundary $\{a,b\}$



$\int_a^b f'(x)dx = \int_a^b \partial_1 f(x)dx = \int_{\{a,b\}} f(s)\langle 1, \nu(s)\rangle ds = f(a)\nu(a) + f(b)\nu(b) = f(b) - f(a)$

Example 2

Let $U = (a_1,b_1) \times (a_2,b_2) \subset \mathbb{R}^2$.



$\int_U \partial_1 f(u)du = \int_{\partial U} f(s)\langle e_i, \nu(s)\rangle ds$

$= \int_{a_2}^{b_2} f(b_1,t)\cdot 1\,dt + \int_{b_1}^{a_1} f(t,b_2)\cdot 0\,dt + \int_{b_2}^{a_2} f(a_1,t)\cdot(-1)\,dt + \int_{a_1}^{b_1} f(t,a_2)\cdot 0\,dt$

$= \int_{a_2}^{b_2} f(a_1,t) + f(b_1,t)\,dt$

Adjoint operators

Adjoint operator

Let $A : X \to Y$ be a linear and continuous operator between two Hilbertspaces. Then there exists a unique operator $B : Y \to X$ such that

$$\langle Ax, y\rangle_Y = \langle x, By\rangle_X$$

B is again a linear and continuous operator. We call B the adjoint operator of A and write $A^* = B$.

The mapping $x \mapsto \langle Ax, y\rangle_Y \in \mathbb{R}$ is linear and continuous.

Riesz: There exists a unique $z =: By \in X$ such that $\langle Ax, y\rangle_Y = \langle x, z\rangle_X$

Linearity

$\langle Ax, y_1 + \alpha y_2\rangle_Y = \langle Ax, y_1\rangle_Y + \alpha\langle Ax, y_2\rangle_Y = \langle x, z_1\rangle_X + \alpha\langle x, z_2\rangle_X = \langle x, z_1 + \alpha z_2\rangle_X$

Adjoint of a matrix

The matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ describes a continuous linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$:

$$A(x) = \mathbf{A}x$$

Adjoint

$$\langle A(x), y\rangle_{\mathbb{R}^m} = \langle \mathbf{A}x, y\rangle_{\mathbb{R}^m} = \langle x, \mathbf{A}^T y\rangle_{\mathbb{R}^n} = \langle x, A^*(y)\rangle_{\mathbb{R}^n}$$

Notice the difference between A and $\mathbf{A}$. In practice A and $\mathbf{A}$ are often identified. The action of linear operators is often abbriviated:

$$A(x) = Ax$$

Adjoint of gradient

Let $\alpha : U \to \mathbb{R}^2$ be a smooth vectorfield on $U \subset \mathbb{R}^2$ and $\tilde{f} \in C_c^\infty(U)$ a test function.

$$\langle \nabla \tilde{f}, \alpha\rangle = \int_U \alpha_1(u)\partial_1 \tilde{f}(u) + \alpha_2(u)\partial_2 \tilde{f}(u)du$$

$$= -\int_U \partial_1\alpha_1(u)\tilde{f}(u)du + \int_{\partial U} \alpha_1(s)\tilde{f}(s)\langle e_1, \nu\rangle ds - \int_U \partial_2\alpha_2(u)\tilde{f}(u)du$$

$$= -\int_U \tilde{f}(u)\,\mathrm{div}\,\alpha(u)du = \langle \tilde{f}, -\mathrm{div}\,\alpha\rangle$$

We say that $-\mathrm{div}$ is *formally* adjoint to ∇.

The gradient is a linear operator but not continuous.

In general one has to carefully choose domain and codomain of operators.

Divergence on manifolds

Smooth vector field

A smooth vectorfield on a compact manifold S is a function

$$V(p) = Dx(x^{-1}(p)) \cdot \begin{pmatrix} \alpha_1(x^{-1}(p)) \\ \alpha_2(x^{-1}(p)) \end{pmatrix}$$

where the coefficient functions $\alpha_i : U \to \mathbb{R}$ are smooth.

Divergence

The divergence of a smooth vectorfield V is the scalar function $\mathrm{div}\,V : S \to \mathbb{R}$ defined via

$$\int_S \langle \nabla f, V\rangle dp = -\int_S f(p)\,\mathrm{div}\,V(p)dp$$

for all test functions $f \in C_c^\infty(S)$.

Divergence in local coordinates

We have seen that it is beneficial to rewrite all kinds of quantities on a surface S as quantities in the better understood parameter domain.

- length of curves
- integrals of functions
- gradient of a function

Our goal is to derive a function $h : U \to \mathbb{R}$ that depends on the vectorfield $\alpha : U \to \mathbb{R}^2$ and satisfies

$$\mathrm{div}\,V(p) = h(x^{-1}(p))$$

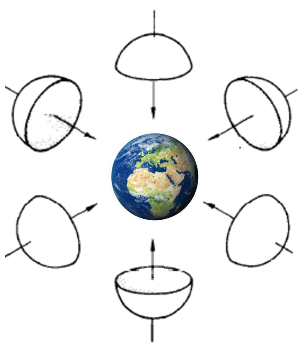
$$\int_S \langle \nabla f, V\rangle dp = -\int_S f(p)\,\mathrm{div}\,V(p)dp = -\int_U \tilde{f}(u)h(u)\sqrt{\det g(u)}du$$

After some work it will turn out that

$$h(u) = \frac{1}{\sqrt{\det g(u)}} \sum_{i=1}^2 \partial_i(\sqrt{\det g(u)}\alpha_i(u))du$$

Main difficulty: Boundaries

The main difficulty arise from the fact that every parameterspace $U_j \subset \mathbb{R}^2$ comes with boundary.



Partition of unity

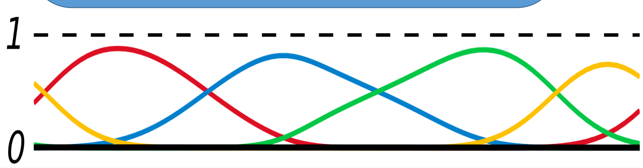
Partition of unity

Let S be a compact manifold and $(U_j, x_j)_{j=1}^N$ such that $\cup_{j=1}^N W_j = \cup_{j=1}^N x_j(U_j) = S$.

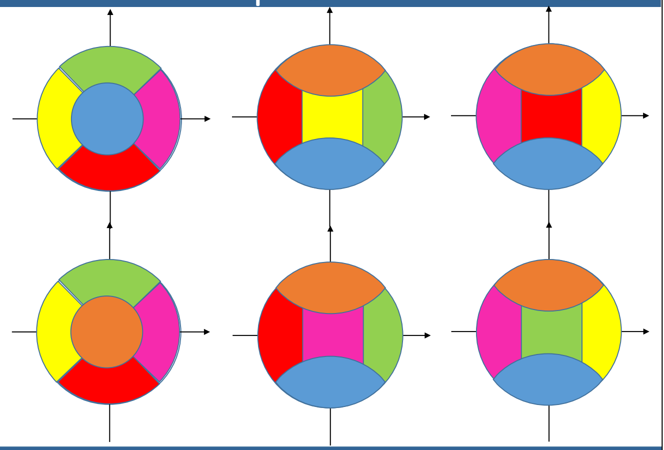
Moreover there exist smooth functions $(\phi_j)_{j=1}^N : S \to \mathbb{R}$ with

- $\mathrm{supp}\,\phi_j \subset\subset W_j$
- $\sum_j \phi_j(p) = 1 \quad \forall p \in S$

The set $\{\phi_j\}_{j=1}^N$ is called (smooth) partition of unity.



Partition of unity on the sphere



Mollifier

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset B_1(0)$ and $\int_{\mathbb{R}^n}\psi(x)dx = 1$.

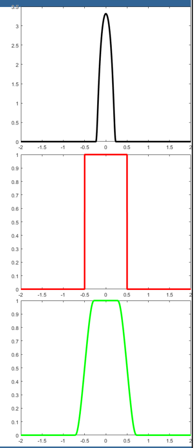
Then $\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi(\frac{x}{\varepsilon})$ satisfies

- $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\psi_\varepsilon \subset B_\varepsilon(0)$
- $\int_{\mathbb{R}^n}\psi_\varepsilon(x)dx = 1$

For $u \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ the functions $u_\varepsilon = u * \psi_\varepsilon$ satisfy

- $u_\varepsilon \in C^\infty(\mathbb{R}^n)$
- $u_\varepsilon \to u$ in $L^p(\mathbb{R}^n)$
- If $\operatorname{supp} u \subset V$, then $\operatorname{supp} u_\varepsilon \subset V_\varepsilon = \{x \in \mathbb{R}^n | d(x,V) < \varepsilon\}$

The functions ψ_ε are called *mollifiers.*



Divergence in local coordinates

Let $V(p) = Dx(u)\alpha(u)$ be a smooth vectorfield on S $(x(u) = p)$ and ϕ_j a partition of unity as on the previous slides.

We define $V_j(p) = V(p)\phi_j(p) = Dx\alpha^j(u)$ and derive

$\operatorname{supp} V^j \subset\subset W_j$
$\operatorname{supp} \alpha^j \subset\subset U_j$
$\sum_j V^j(p) = V(p)$

$$\int_S \langle\nabla f, V\rangle dp = \sum_j \sum_{i=1}^2 \int_{U_j} (\partial_i \tilde f(u))\alpha_i^j(u)\sqrt{\det g(u)}du$$

$$= -\sum_j \sum_{i=1}^2 \int_{U_j} \tilde f(u)\partial_i(\alpha_i^j(u)\sqrt{\det g(u)})du$$

$$= -\sum_{i=1}^2 \sum_j \int_{W_j} f(p)\langle\nabla(\alpha_i^j\sqrt{\det g}\circ x^{-1}), \partial_i x\rangle \frac{1}{\sqrt{\det g}}dp$$

$$= \int_S f(p)\left(-\sum_{i=1}^2 \langle\nabla(\alpha_i\sqrt{\det g}\circ x^{-1}), \partial_i x\rangle \frac{1}{\sqrt{\det g}}\right)dp$$

$$= \int_U \tilde f(u)\left(-\sum_{i=1}^2 (\partial_i\alpha_i\sqrt{\det g})\frac{1}{\sqrt{\det g}}\right)\sqrt{\det g}\,du$$

$$= \int_U \tilde f(u)h(u)\sqrt{\det g}\,du$$

Functionals

Functional

Let $\mathcal{F}$ be some space of functions. A **functional** is a mapping

$$E : \mathcal{F} \to \mathbb{R}$$

In Computer Vision functionals are frequently called **energies.**

Example:

Let (x, U) be a parametrization of a surface S. We consider the space of differentiable curves on S that connect $p = x(u)$ and $q = x(v)$.

$$\mathcal{F} = \{\gamma \in C^1((a,b),U) : \gamma(a) = u,\ \gamma(b) = v\}$$

Then the mapping $E(\gamma) = L(x(\gamma)) = \int_a^b \sqrt{\langle\dot\gamma(x), \dot\gamma(x)\rangle_{g(\gamma(x))}}dx$ is a functional on $\mathcal{F}$.

Motivation

Often one is interested in the minimizers of Functionals.

$$f^* = \operatorname{argmin}_{f \in F} E(f)$$

Let us recap how we found minimizers of a function $F : \mathbb{R}^n \to \mathbb{R}$

As a first step one is looking for a point x^* such that

$$\nabla F(x^*) = 0$$

In other words: The directional derivatives

$$DF(x^*)[v] = \lim_{\varepsilon\to 0}\frac{F(x^*+\varepsilon v) - F(x^*)}{\varepsilon} = 0$$

vanish for all directions $v \in \mathbb{R}^n$

Test directions

We will use a similar approach when minimizing functionals but have to be careful with the allowed directions v.

Consider the following problem

$$\min\{E(u) : u \in \mathcal{F}\}, \text{ with } \mathcal{F} = \{u \in C^\infty([a,b]), u(a) = \alpha,\ u(b) = \beta\}$$

We will only test directions $v \in C_c^\infty((a,b])$. That way we are sure that we only compare energies of members of $\mathcal{F}$:

$$u \in \mathcal{F} \Rightarrow u + \varepsilon v \in \mathcal{F} \quad \forall v \in C_c^\infty((a,b))$$

Euler Lagrange

A minimizer u^* of

$$E(u) = \int_a^b f(u(x), u'(x))dx,\ u \in \mathcal{F}, f \in C^2(\mathbb{R}\times\mathbb{R})$$

$$\mathcal{F} = \{u \in C^\infty([a,b]), u(a) = \alpha,\ u(b) = \beta\}$$

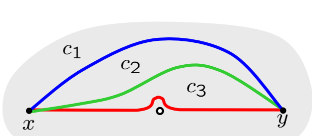
must satisfy

$$\frac{d}{dx}[\partial_2 f(u^*, (u^*)')] = \partial_1 f(u^*, (u^*)')$$

This is the Euler Lagrange equation of $E(u)$.

Remark

- A minimizer must not exist



Proof

First we define $\Phi(\varepsilon) = E(u^* + \varepsilon v)$ and observe

$$\lim_{\varepsilon\to 0}\frac{E(u^*+\varepsilon v) - E(u)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\Phi(\varepsilon) - \Phi(0)}{\varepsilon} = \Phi'(0)$$

and

$$\Phi'(0) = \int_a^b \frac{d}{d\varepsilon}|_{\varepsilon=0} f(u + \varepsilon v, u' + \varepsilon v')dx$$

$$= \int_a^b \partial_1 f(u, u')v(x) + \partial_2 f(u, u')v'(x)dx$$

with partial integration and $v \in C_c^\infty([a,b])$

$$= \int_a^b (\partial_1 f(u, u') - \frac{d}{dx}\partial_2 f(u, u'))v(x)dx = DE(u)[v] = \frac{\partial E}{\partial u}(u)[v]$$

Fund. Lemma of Calc. Var

Fundamental Lemma of the calculus of variations

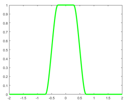
Let $U \subset \mathbb{R}^n$ be open and $u \in C^\infty(U)$ be such that

$$\int_U u(x)v(x) = 0 \quad \forall v \in C_c^\infty(U)$$

then $u = 0$.

Proof

Assume $u(y) \neq 0$ for some $y \in U$. For instance $u(y) > 0$.
Then u is strictly positive in a neighborhood $B_r(y)$ (due to continuity).
There is a positive function $v \in C_c^\infty(B_r(y)) \subset C_c^\infty(U)$ with $v(y) = 1$.
(For instance $v = \mathbf{1}_{B_{r/2}(y)} * \psi_{r/4}$)



$$\Rightarrow \int_U u(x)v(x) > 0$$

Generalization

Let $U \subset \mathbb{R}^n$. A minimizer u^* of

$$E(u) = \int_U f(u(x), \nabla u(x))dx,\ u \in \mathcal{F}, f \in C^2(\mathbb{R} \times \mathbb{R}^n)$$

$$\mathcal{F} = \{u \in C^\infty(U, \mathbb{R}), u|_{\partial U} = g\}$$

must satisfy

$$\operatorname{div}[\nabla_2 f(u^*, \nabla u^*)] = \partial_1 f(u^*, \nabla u^*)$$

Example (Dirichlet energy)

$$E(u) = \int_U \|\nabla u\|^2 dx \qquad f(u,\xi) = \|\xi\|^2$$

$$\partial_1 f(u,\xi) = 0, \quad \nabla_2 f(u,\xi) = 2\xi \qquad \Rightarrow \operatorname{div}(2\nabla u^*) = 0$$

Generalizations 2

A minimizer $u^* \in C^2([a,b], \mathbb{R}^n)$ of

$$E(u) = \int_a^b f(u_1(x), \dots, u_n(x), u_1'(x), \dots, u_n'(x))dx,\ u \in \mathcal{F}, f \in C^2(\underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{2n})$$

$$\mathcal{F} = \{u \in C^1([a,b], \mathbb{R}^n), u(a) = \alpha,\ u(b) = \beta\}$$

must satisfy

$$\frac{d}{dx}[\partial_{i+n} f(u^*, (u^*)')] = \partial_i f(u^*, (u^*)') \quad \forall i = 1, \dots, n$$