Analysis of 3D Shapes (IN2238)

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11. Geodesics and Curvature

Covariant Derivative

Christoffel Symbols

Given a coordinate map $x\colon U \to M$ of the n-dimensional manifold $M \subset \mathbb{R}^{n+1}$, the canonical Riemannian metric is given as

$$g\colon U \to \mathbb{R}^{n\times n} \qquad g_{ij}(u) = \langle \partial_i x(u), \partial_j x(u)\rangle$$

While the first derivatives $\partial_i x(u)$ lie in the n-dimensional vector space $T_{x(u)}M$, the second derivatives might contain a normal component, *i.e.*,

$$\partial_{ij}x(u) = \sum_{k=1}^{n} \Gamma_{ij}^k(u)\partial_k x(u) + \alpha_{ij}(u)N(u)$$

The n^3 scalar functions $\Gamma_{ij}^k\colon U \to \mathbb{R}$ are called **Christoffel symbols**.
They are symmetric in i and j, *i.e.*, $\Gamma_{ij}^k = \Gamma_{ji}^k$. (Why?)

Christoffel Symbols and Metric

Using $\partial_i g_{j\ell}(u) = \langle \partial_{ij}x(u), \partial_\ell x(u)\rangle + \langle \partial_{i\ell}x(u), \partial_j x(u)\rangle$, we obtain

$$\begin{aligned}\tilde{\Gamma}_{\ell ij}(u) :=& \frac{1}{2}[\partial_i g_{j\ell}(u) + \partial_j g_{\ell i}(u) - \partial_\ell g_{ij}(u)]\\ =& \frac{1}{2}[\langle \partial_{ij}x(u), \partial_\ell x(u)\rangle + \langle \partial_{i\ell}\mathbf{x}(\mathbf{u}), \partial_j\mathbf{x}(\mathbf{u})\rangle + \langle \partial_{j\ell}\mathbf{x}(\mathbf{u}), \partial_i\mathbf{x}(\mathbf{u})\rangle +\\ & \langle \partial_{ji}x(u), \partial_\ell x(u)\rangle - \langle \partial_{\ell i}\mathbf{x}(\mathbf{u}), \partial_j\mathbf{x}(\mathbf{u})\rangle - \langle \partial_{\ell j}\mathbf{x}(\mathbf{u}), \partial_i\mathbf{x}(\mathbf{u})\rangle]\\ =& \langle \partial_{ij}x(u), \partial_\ell x(u)\rangle = \sum_{k=1}^{n}\Gamma_{ij}^k(u)g_{k\ell}(u)\end{aligned}$$

If we use the notation $g^{ij}(u) := (g(u)^{-1})_{ij}$, we obtain

$$\sum_{\ell=1}^{n} g^{k\ell}(u)\tilde{\Gamma}_{\ell ij}(u) = \sum_{k'=1}^{n}\sum_{\ell=1}^{n} g^{k\ell}(u)g_{\ell k'}(u)\Gamma_{ij}^{k'}(u) = \Gamma_{ij}^k(u)$$

Christoffel Symbols are Intrinsic

In summary, we have

$$\partial_{ij}x = \sum_{k=1}^{n}\Gamma_{ij}^k\partial_k x + \alpha_{ij}N$$

with the **intrinsic** Christoffel symbols

$$\Gamma_{ij}^k = \sum_{\ell=1}^{n}\frac{1}{2}g^{kl}[\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}]$$

The expression $\sum_{k=1}^n \Gamma_{ij}^k\partial_k x$ can also be seen as an intrinsic derivative of the vector field $\partial_j x$ in the direction of $\partial_i x$.

This derivative is called **covariant derivative**.

Example: Sphere

Given the coordinate map

$$x\colon \left]-\frac{\pi}{3},\frac{\pi}{3}\right[\times \left]-\frac{\pi}{3},\frac{\pi}{3}\right[\to \mathbb{S}^2$$

$$(\alpha_1,\alpha_2) \mapsto \begin{pmatrix}\cos(\alpha_1)\cos(\alpha_2)\\ \sin(\alpha_1)\cos(\alpha_2)\\ \sin(\alpha_2)\end{pmatrix}$$

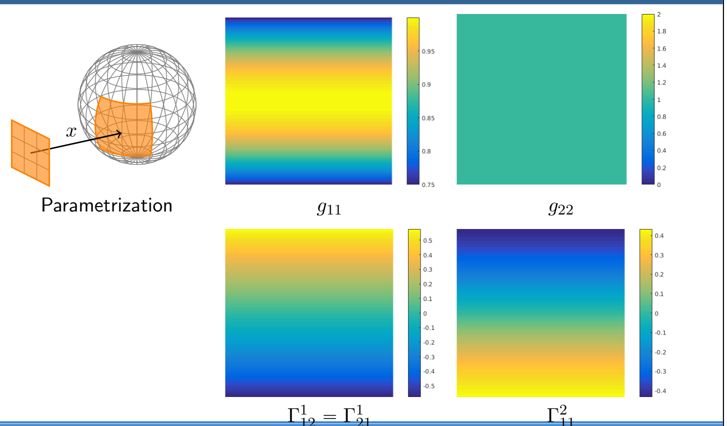
we obtain the Riemannian metric

$$g(\alpha_1,\alpha_2) = \begin{pmatrix}\cos(\alpha_2)^2 & 0\\ 0 & 1\end{pmatrix}$$

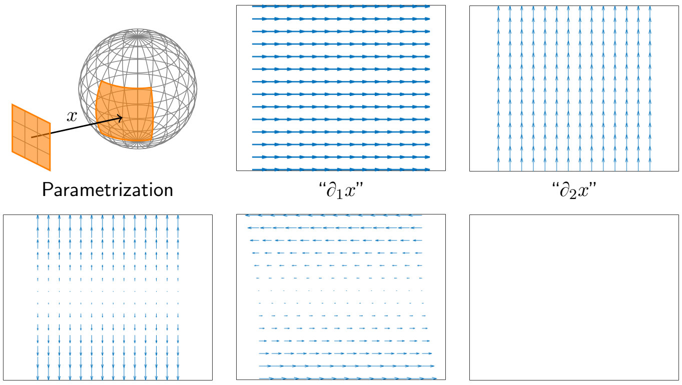
and the Christoffel symbols

$$\Gamma^1(\alpha_1,\alpha_2) = -\frac{\sin(2\alpha_2)}{2\cos(\alpha_2)^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \Gamma^2(\alpha_1,\alpha_2) = \frac{\sin(2\alpha_2)}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

Example: Christoffel Symbols



Example: Covariant Derivative



Parametrization | "$\partial_1 x$" | "$\partial_2 x$"

"$\nabla_{\partial_1 x}\partial_1 x$" | "$\nabla_{\partial_1 x}\partial_2 x$" = "$\nabla_{\partial_2 x}\partial_1 x$" | "$\nabla_{\partial_2 x}\partial_2 x$"

Covariant Derivative

Given a coordinate map $x\colon U \to M$ of the n-dimensional manifold $M \subset \mathbb{R}^{n+1}$, and two vector fields Y and Z represented as $(p = x(u))$

$$Y(p) = \sum_{i=1}^{n} y_i(u)\partial_i x(u) \qquad Z(p) = \sum_{j=1}^{n} z_j(u)\partial_j x(u),$$

the **covariant derivative** $\nabla_Z Y$ is a vector field that can be represented as

$$\left[\nabla_{\partial_j x}\partial_i x\right](p) = \sum_{k=1}^{n} \Gamma_{ij}^{k}(u)\partial_k x(u)$$

$$\left[\nabla_{\partial_j x} Y\right](p) = \sum_{i=1}^{n} y_i(u)\nabla_{\partial_j x}\partial_i x(p) + \partial_j y_i(u)\partial_i x(u) \qquad \text{(product rule in } Y\text{)}$$

$$\left[\nabla_Z Y\right](p) = \sum_{j=1}^{n} z_j(u)\cdot\nabla_{\partial_j x} Y(p) \qquad \text{(linearity in } Z\text{)}$$

Extrinsic Formulation

$\nabla_Z Y$ can be formulated in a simpler manner if Y and Z can be extended to the ambient space $\mathbb{R}^{n+1}$ of M. To this end let

$$\tilde{Y}, \tilde{Z}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

with $\tilde{Y}|M = Y$ and $\tilde{Z}|M = Z$.

Then, we have for every $p \in M$

$$\nabla_Z Y(p) = \pi_{T_pM}\left(D\tilde{Y}(p)\cdot\tilde{Z}(p)\right),$$

where

$$\pi_{T_pM}\colon \mathbb{R}^{n+1} \to T_pM$$

is the orthogonal projection of the ambient space $\mathbb{R}^{n+1}$ onto T_pM.

Geodesics

Shortest Path in Local Coordinates

Given a coordinate map $x\colon U \to M$ of the n-dimensional manifold M, we like to find the shortest path $\gamma\colon [0,1] \to U$ that connects two points $u_0, u_1 \in U$.

The length of γ is induced by the Riemannian metric $g\colon U \to \mathbb{R}^{n\times n}$ via

$$\text{length}(\gamma) = \int_0^1 \langle\dot\gamma(t), g(\gamma(t))\cdot\dot\gamma(t)\rangle^{\frac{1}{2}}\,\mathrm{d}t$$

It is often easier to consider the following energy function instead

$$E(\gamma) = \left[\int_0^1 \langle\dot\gamma(t), g(\gamma(t))\cdot\dot\gamma(t)\rangle\,\mathrm{d}t\right]^{\frac{1}{2}}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\text{length}(\gamma) \leqslant E(\gamma)$$

with equality iff $\|\dot\gamma\|_g \equiv \text{const}$, *i.e.*, γ is uniformly parametrized.

Geodesics

Let us select the two minimizers $\gamma^* \in \operatorname{argmin}\text{length}(\cdot)$ and $\hat\gamma \in \operatorname{argmin} E(\cdot)$. Further we assume that $\bar\gamma^*$ is a uniform re-parametrization of γ^*.

Then we have

$$\text{length}(\gamma^*) = \text{length}(\bar\gamma^*) = E(\bar\gamma^*) \geqslant E(\hat\gamma) \geqslant \text{length}(\hat\gamma) \geqslant \text{length}(\gamma^*).$$

Therefore, we know

- Every minimizer of E minimizes length $[\text{length}(\hat\gamma) = \text{length}(\gamma^*)]$
- The minimum of E is the minimal length $[\text{length}(\hat\gamma) = E(\hat\gamma)]$
- The minimizer of E is uniformly parametrized $[\text{length}(\hat\gamma) = E(\hat\gamma)]$

Minimizing E provides us with a uniformly parametrized shortest path between two points. Every local minimum of E is called **geodesic**.

Geodesic Equation

Given two points $u_0, u_1 \in U$, a geodesic $\gamma = (\gamma_1, \ldots, \gamma_n)\colon [0,1] \to U$ that connects these points minimizes

$$E(\gamma_1, \ldots, \gamma_n) := \int_0^1 \sum_{i,j=1}^{n} g_{ij}(\gamma(t))\cdot\dot\gamma^i(t)\dot\gamma^j(t)\mathrm{d}t$$

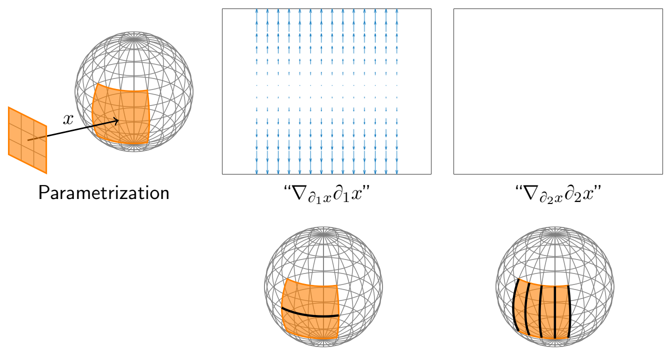
The **Euler-Lagrange equation** is

$$0 = \frac{\partial E}{\partial\gamma_k} = \sum_{i,j=1}^{n} \partial_k g_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[2\sum_{i=1}^{n} g_{ik}(\gamma(t))\dot\gamma^i(t)\right]$$

$$\ddot\gamma^k = -\left\langle\dot\gamma, \Gamma^k\dot\gamma\right\rangle$$

and can therefore be presented with respect to the Christoffel symbols.

Example: Geodesics



Parametrization | "$\nabla_{\partial_1 x}\partial_1 x$" | "$\nabla_{\partial_2 x}\partial_2 x$"

Equator | Meridians

Covariant Derivative along Curves

Given a curve $\gamma\colon (0,L) \to U$ and a vector field X along the curve $c = x \circ \gamma$, we would like to define

$$\frac{\nabla}{\mathrm{dt}} X := \nabla_{\dot c} X$$

To this end let $Y = \sum_{i=1}^n y^i \partial_i x$ be a vector field on M that coincides along c with X. Further let $Z = \sum_{i=1}^n z^i \partial_i x$ be a vector field that coincides along c with $\dot c$. Then we have ($p = x(u) = c(\tau)$)

$$\nabla_Z Y(p) = \sum_{k=1}^n \left[\frac{\mathrm{d}}{\mathrm{dt}} \left(y^k \circ \gamma(t) \right)\Big|_{t=\tau} + \sum_{i,j=1}^n y^i(u) \Gamma_{ij}^k(u) z^j(u) \right] \partial_k x(u)$$

If we restrict this vector field to a vector field along the curve, it only depends on X and c, but not on the extension of Y and Z. Thus, $\frac{\nabla}{\mathrm{dt}}$ is well defined.

Geodesic Equation in Terms of the Covariant Derivative

Given a geodesic $c\colon (0,1) \to M$, we have for $\dot c = \sum_{i=1}^n \dot\gamma^i \partial_i x$

$$\begin{aligned} \frac{\nabla}{\mathrm{dt}} \dot c &= \sum_{k=1}^n \left[\ddot\gamma^k + \sum_{i,j=1}^n \dot\gamma^i \Gamma_{ij}^k \dot\gamma^j \right] \partial_k x \\ &= \sum_{k=1}^n \left[\ddot\gamma^k + \left\langle \dot\gamma, \Gamma^k \dot\gamma \right\rangle \right] \partial_k x = 0 \end{aligned}$$

The geodesic equation can therefore be written as

$$\frac{\nabla}{\mathrm{dt}} \dot c = 0$$

Since $\frac{\nabla}{\mathrm{dt}} \dot c$ measures how different a curve c is from a geodesic we can use it to define the **geodesic curvature** of a curve.

Geodesic curvature

Given a curve $c\colon (0,L) \to \mathbb{R}^2$ parametrized by arc-length ($\|\dot c\| \equiv 1$), the curvature $\kappa(t)$ at $c(t)$ can be computed via

$$\kappa(t) = \frac{\det(\dot c(t), \ddot c(t))}{\|\dot c(t)\|^3} = \det(\dot c(t), \ddot c(t))$$

Given a curve $c\colon (0,L) \to M$ in the 2D manifold M that is parametrized by arc-length, we can compute the **geodesic curvature** $\kappa_g(t)$ by replacing $\ddot c$ with $\frac{\nabla}{\mathrm{dt}} \dot c$ and obtain

$$\kappa_g(t) = \det \left(\dot c(t), \frac{\nabla}{\mathrm{dt}} \dot c(t) \right)$$

The geodesic curvature is 0 for geodesics and can therefore be understood as an intrinsic reformulation of the classical curvature of curves.

Second Fundamental Form

Gauss Map

Given a 2D manifold $M \subset \mathbb{R}^3$, we call a smooth mapping

$$N\colon M \to \mathbb{S}^2 \qquad \forall p \in M\colon N(p) \perp T_p M$$

its **Gauss map**. For every 3D shape there exists such a mapping. (Why?)

If $x\colon U \to M$ is a coordinate mapping, we can always define a local Gauss map via

$$\begin{aligned} N\colon M &\to \mathbb{S}^2 \\ p &\mapsto \frac{\partial_1 x(u) \times \partial_2 x(u)}{\|\partial_1 x(u) \times \partial_2 x(u)\|} \qquad \text{for } u = x^{-1}(p) \end{aligned}$$

If $M = f^{-1}(c)$ is given implicitly via a function $f\colon \mathbb{R}^3 \to \mathbb{R}$, the Gauss map is given via $N(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$.

Shape Operator

Given a 2D manifold $M \subset \mathbb{R}^3$ together with its Gauss map $N\colon M \to \mathbb{S}^2$, we call its differential the **shape operator** or **Weingarten mapping** S

$$\begin{aligned} S_p\colon T_p M &\to T_{N(p)} \mathbb{S}^2 \\ v &\mapsto DN(p)[v] \end{aligned}$$

Since $T_{N(p)} \mathbb{S}^2 = N(p)^\perp = T_p M$, $S_p\colon T_p M \to T_p M$ is an endomorphism.

If we choose a basis of $T_p M$, we would obtain a 2×2 matrix, but this matrix would depend on the chosen basis. Nonetheless, the eigenvalues of these matrices would remain the same.

The goal is to show that S_p can be put in diagonal form and that both eigenvalues are real.

Self-Adjointness of the Shape Operator

We know (Linear Algebra) that self-adjoint endomorphisms are diagonalizable with real eigenvalues. Therefore, we have to prove that

$$\langle v_1, S_p(v_2) \rangle = \langle S_p(v_1), v_2 \rangle \qquad \text{for all } v_1, v_2 \in T_p M$$

If v_1 and v_2 are co-linear this is obvious. If they are not co-linear, one can find a local coordinate map $x\colon U \to M$ with $x(0) = p$ and $v_i = \partial_i x(0)$.

Using $\langle N \circ x(u), \partial_i x(u) \rangle \equiv 0$ leads to

$$\begin{aligned} 0 &= \partial_1 \langle N \circ x(u), \partial_2 x(u) \rangle|_{u=0} = \langle S_p(v_1), v_2 \rangle + \langle N(p), \partial_{12} x(0) \rangle \\ 0 &= \partial_2 \langle N \circ x(u), \partial_1 x(u) \rangle|_{u=0} = \langle S_p(v_2), v_1 \rangle + \langle N(p), \partial_{21} x(0) \rangle \end{aligned}$$

which proves the self-adjointness of the shape operator.

Principal Curvatures

The two eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of S_p are called **principal curvatures** and corresponding eigenvectors $v_1(p)$ and $v_2(p)$ are called **principal curvature directions**.

Note that $\kappa_g(p)$ along the geodesic c_i corresponding to $v_i(p)$ is 0 and the curvature of this curve coincides with $\kappa_i(p)$. In that sense, we can think of the principal curvatures as natural generalizations of the planar curvature.

We can derive two other curvatures from the principal curvatures:

$$\begin{aligned} H(p) &:= \frac{\kappa_1(p) + \kappa_2(p)}{2} &&= \frac{1}{2} \operatorname{tr}(\mathcal{M}) && \text{(mean curvature)} \\ K(p) &:= \kappa_1(p) \cdot \kappa_2(p) &&= \det(\mathcal{M}) && \text{(Gauss curvature)} \end{aligned}$$

given a representing matrix $\mathcal{M}$ of S_p.

Second Fundamental Form

Given the shape operator $S_p \colon T_pM \to T_pM$, we can define the **Second Fundamental Form**

$$\mathbb{I} \colon T_pM \times T_pM \to \mathbb{R} \qquad (v_1, v_2) \mapsto \langle S_p v_1, v_2 \rangle$$

This means, we have

$$\partial_{ij} x = \sum_{k=1}^{n} \Gamma_{ij}^{k} \partial_k x - \mathbb{I}(\partial_i x, \partial_j x) \cdot N$$

and the second fundamental form can be computed via

$$\mathbb{I}(\partial_i x, \partial_j x) = -\langle \partial_{ij} x, N \rangle .$$

Shape Operator in Local Coordinates

Any coordinate map $x \colon U \to M$ provides for a base $\{\partial_1 x(u), \ldots, \partial_n x(u)\}$ of T_pM for $p = x(u)$. In this base, the shape operator S_p can be written as

$$S_p(\partial_j x(u)) = \sum_{i=1}^{n} \mathcal{M}_j^i \partial_i x(u)$$

This means, we have

$$-\mathbb{I}(\partial_j x, \partial_k x) = \langle S_p(\partial_j x), \partial_k x \rangle = \sum_{i=1}^{n} \left\langle \mathcal{M}_j^i \partial_i x, \partial_k x \right\rangle = \sum_{i=1}^{n} g_{ki} \mathcal{M}_j^i$$

In other words the representating matrix $\mathcal{M}$ of S_p satisfies the **Weingarten equations**

$$\mathcal{M} = -g^{-1} \cdot \mathbb{I}$$