

Analysis of 3D Shapes (IN2238)

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11. Geodesics and Curvature

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Covariant Derivative

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Christoffel Symbols

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold $M \subset \mathbb{R}^{n+1}$, the canonical Riemannian metric is given as

$$g: U \rightarrow \mathbb{R}^{n \times n} \qquad g_{ij}(u) = \langle \partial_i x(u), \partial_j x(u) \rangle$$

While the first derivatives $\partial_i x(u)$ lie in the n -dimensional vector space $T_{x(u)}M$, the second derivatives might contain a normal component, *i.e.*,

$$\partial_{ij} x(u) = \sum_{k=1}^n \Gamma_{ij}^k(u) \partial_k x(u) + \alpha_{ij}(u) N(u)$$

The n^3 scalar functions $\Gamma_{ij}^k: U \rightarrow \mathbb{R}$ are called **Christoffel symbols**.

They are symmetric in i and j , *i.e.*, $\Gamma_{ij}^k = \Gamma_{ji}^k$. (Why?)

Christoffel Symbols and Metric

Using $\partial_i g_{j\ell}(u) = \langle \partial_{ij}x(u), \partial_\ell x(u) \rangle + \langle \partial_{i\ell}x(u), \partial_j x(u) \rangle$, we obtain

$$\begin{aligned}\tilde{\Gamma}_{\ell ij}(u) &:= \frac{1}{2}[\partial_i g_{j\ell}(u) + \partial_j g_{\ell i}(u) - \partial_\ell g_{ij}(u)] \\ &= \frac{1}{2}[\langle \partial_{ij}x(u), \partial_\ell x(u) \rangle + \langle \partial_{i\ell}\mathbf{x}(u), \partial_j \mathbf{x}(u) \rangle + \langle \partial_{j\ell}\mathbf{x}(u), \partial_i \mathbf{x}(u) \rangle + \\ &\quad \langle \partial_{ji}x(u), \partial_\ell x(u) \rangle - \langle \partial_{\ell i}\mathbf{x}(u), \partial_j \mathbf{x}(u) \rangle - \langle \partial_{\ell j}\mathbf{x}(u), \partial_i \mathbf{x}(u) \rangle] \\ &= \langle \partial_{ij}x(u), \partial_\ell x(u) \rangle = \sum_{k=1}^n \Gamma_{ij}^k(u) g_{k\ell}(u)\end{aligned}$$

If we use the notation $g^{ij}(u) := (g(u)^{-1})_{ij}$, we obtain

$$\sum_{\ell=1}^n g^{k\ell}(u) \tilde{\Gamma}_{\ell ij}(u) = \sum_{k'=1}^n \sum_{\ell=1}^n g^{k\ell}(u) g_{\ell k'}(u) \Gamma_{ij}^{k'}(u) = \Gamma_{ij}^k(u)$$

Christoffel Symbols are Intrinsic

In summary, we have

$$\partial_{ij}x = \sum_{k=1}^n \Gamma_{ij}^k \partial_k x + \alpha_{ij}N$$

with the **intrinsic** Christoffel symbols

$$\Gamma_{ij}^k = \sum_{\ell=1}^n \frac{1}{2} g^{k\ell} [\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}]$$

The expression $\sum_{k=1}^n \Gamma_{ij}^k \partial_k x$ can also be seen as an intrinsic derivative of the vector field $\partial_j x$ in the direction of $\partial_i x$.

This derivative is called **covariant derivative**.

Example: Sphere

Given the coordinate map

$$x: \left] -\frac{\pi}{3}, \frac{\pi}{3} \right[\times \left] -\frac{\pi}{3}, \frac{\pi}{3} \right[\rightarrow \mathbb{S}^2$$
$$(\alpha_1, \alpha_2) \mapsto \begin{pmatrix} \cos(\alpha_1) \cos(\alpha_2) \\ \sin(\alpha_1) \cos(\alpha_2) \\ \sin(\alpha_2) \end{pmatrix}$$

we obtain the Riemannian metric

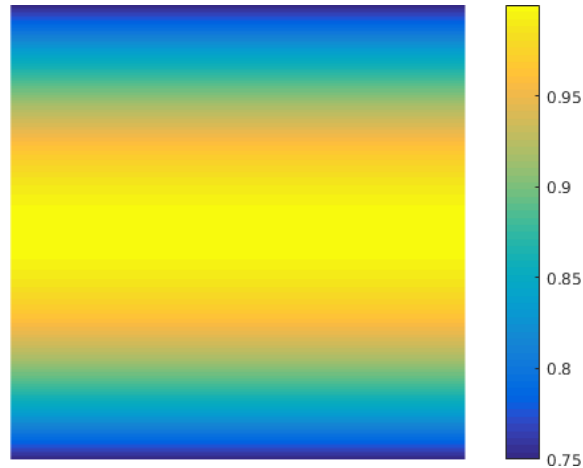
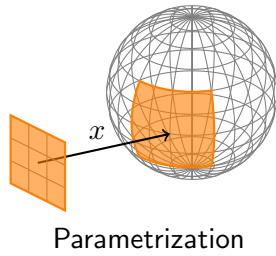
$$g(\alpha_1, \alpha_2) = \begin{pmatrix} \cos(\alpha_2)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and the Christoffel symbols

$$\Gamma^1(\alpha_1, \alpha_2) = -\frac{\sin(2\alpha_2)}{2\cos(\alpha_2)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma^2(\alpha_1, \alpha_2) = \frac{\sin(2\alpha_2)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

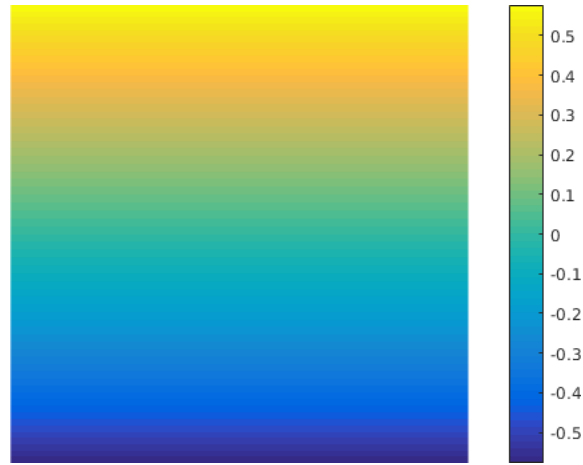
Example: Christoffel Symbols



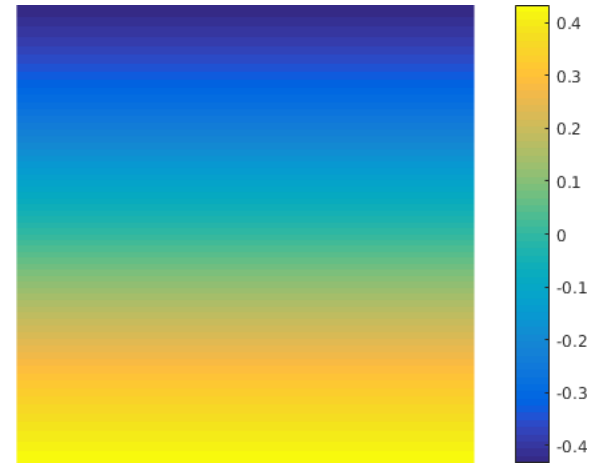
g_{11}



g_{22}

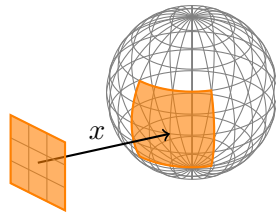


$\Gamma_{12}^1 = \Gamma_{21}^1$

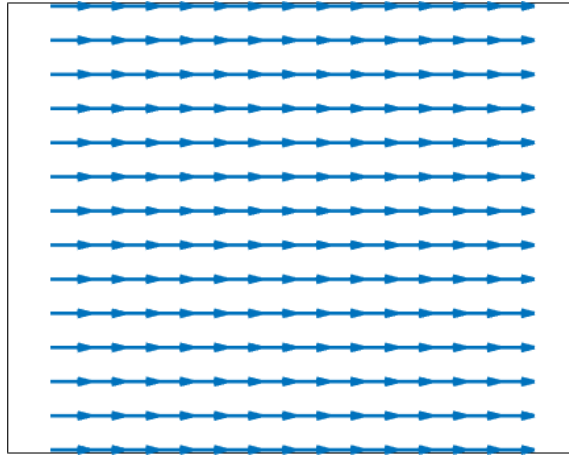


Γ_{11}^2

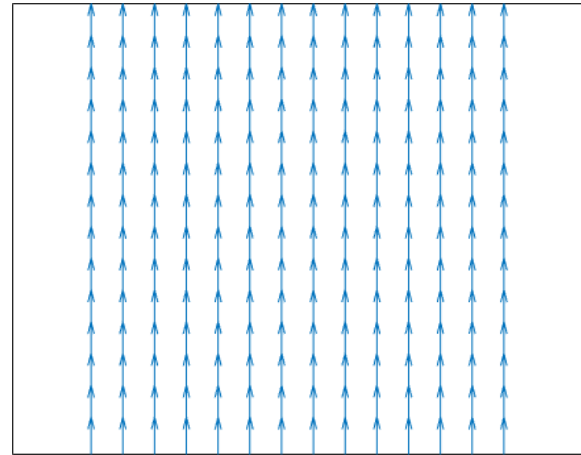
Example: Covariant Derivative



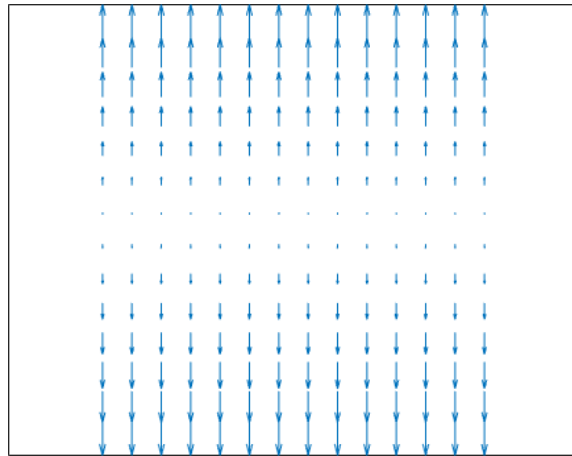
Parametrization



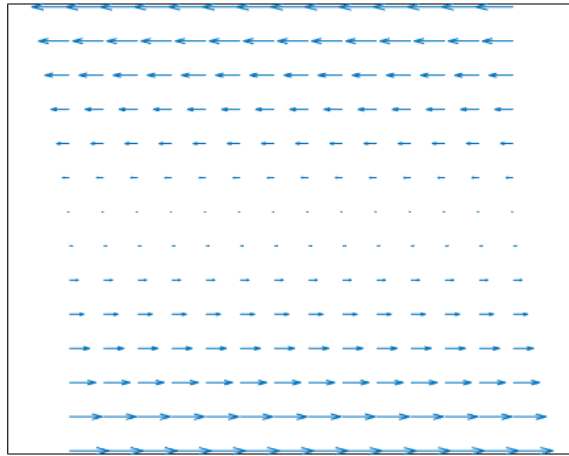
" $\partial_1 x$ "



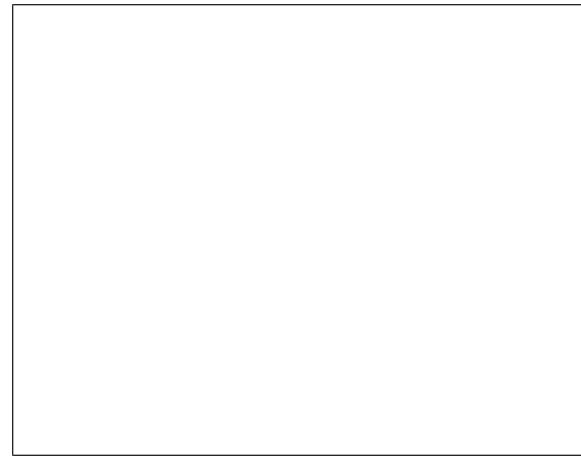
" $\partial_2 x$ "



" $\nabla_{\partial_1 x} \partial_1 x$ "



" $\nabla_{\partial_1 x} \partial_2 x = \nabla_{\partial_2 x} \partial_1 x$ "



" $\nabla_{\partial_2 x} \partial_2 x$ "

Covariant Derivative

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold $M \subset \mathbb{R}^{n+1}$, and two vector fields Y and Z represented as ($p = x(u)$)

$$Y(p) = \sum_{i=1}^n y_i(u) \partial_i x(u)$$

$$Z(p) = \sum_{j=1}^n z_j(u) \partial_j x(u),$$

the **covariant derivative** $\nabla_Z Y$ is a vector field that can be represented as

$$[\nabla_{\partial_j x} \partial_i x](p) = \sum_{k=1}^n \Gamma_{ij}^k(u) \partial_k x(u)$$

$$[\nabla_{\partial_j x} Y](p) = \sum_{i=1}^n y_i(u) \nabla_{\partial_j x} \partial_i x(p) + \partial_j y_i(u) \partial_i x(u) \quad (\text{product rule in } Y)$$

$$[\nabla_Z Y](p) = \sum_{j=1}^n z_j(u) \cdot \nabla_{\partial_j x} Y(p) \quad (\text{linearity in } Z)$$

Extrinsic Formulation

$\nabla_Z Y$ can be formulated in a simpler manner if Y and Z can be extended to the ambient space \mathbb{R}^{n+1} of M . To this end let

$$\tilde{Y}, \tilde{Z}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

with $\tilde{Y}|_M = Y$ and $\tilde{Z}|_M = Z$.

Then, we have for every $p \in M$

$$\nabla_Z Y(p) = \pi_{T_p M} \left(D\tilde{Y}(p) \cdot \tilde{Z}(p) \right),$$

where

$$\pi_{T_p M}: \mathbb{R}^{n+1} \rightarrow T_p M$$

is the orthogonal projection of the ambient space \mathbb{R}^{n+1} onto $T_p M$.

Shortest Path in Local Coordinates

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold M , we like to find the shortest path $\gamma: [0, 1] \rightarrow U$ that connects two points $u_0, u_1 \in U$.

The length of γ is induced by the Riemannian metric $g: U \rightarrow \mathbb{R}^{n \times n}$ via

$$\text{length}(\gamma) = \int_0^1 \langle \dot{\gamma}(t), g(\gamma(t)) \cdot \dot{\gamma}(t) \rangle^{\frac{1}{2}} dt$$

It is often easier to consider the following energy function instead

$$E(\gamma) = \left[\int_0^1 \langle \dot{\gamma}(t), g(\gamma(t)) \cdot \dot{\gamma}(t) \rangle dt \right]^{\frac{1}{2}}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\text{length}(\gamma) \leq E(\gamma)$$

with equality iff $\|\dot{\gamma}\|_g \equiv \text{const}$, i.e., γ is uniformly parametrized.

Geodesics

Let us select the two minimizers $\gamma^* \in \operatorname{argmin} \operatorname{length}(\cdot)$ and $\hat{\gamma} \in \operatorname{argmin} E(\cdot)$. Further we assume that $\bar{\gamma}^*$ is a uniform re-parametrization of γ^* .

Then we have

$$\operatorname{length}(\gamma^*) = \operatorname{length}(\bar{\gamma}^*) = E(\bar{\gamma}^*) \geq E(\hat{\gamma}) \geq \operatorname{length}(\hat{\gamma}) \geq \operatorname{length}(\gamma^*).$$

Therefore, we know

Every minimizer of E minimizes length	[length($\hat{\gamma}$) = length(γ^*)]
The minimum of E is the minimal length	[length($\hat{\gamma}$) = $E(\hat{\gamma})$]
The minimizer of E is uniformly parametrized	[length($\hat{\gamma}$) = $E(\hat{\gamma})$]

Minimizing E provides us with a uniformly parametrized shortest path between two points. Every local minimum of E is called **geodesic**.

Geodesic Equation

Given two points $u_0, u_1 \in U$, a geodesic $\gamma = (\gamma_1, \dots, \gamma_n): [0, 1] \rightarrow U$ that connects these points minimizes

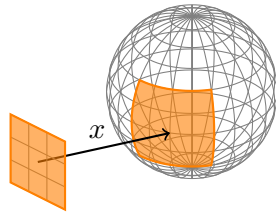
$$E(\gamma_1, \dots, \gamma_n) := \int_0^1 \sum_{i,j=1}^n g_{ij}(\gamma(t)) \cdot \dot{\gamma}^i(t) \dot{\gamma}^j(t) dt$$

The **Euler-Lagrange equation** is

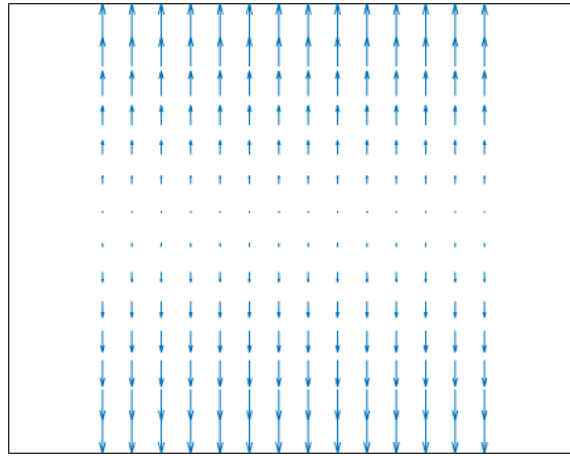
$$0 = \frac{\partial E}{\partial \gamma_k} = \sum_{i,j=1}^n \partial_k g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) - \frac{d}{dt} \left[2 \sum_{i=1}^n g_{ik}(\gamma(t)) \dot{\gamma}^i(t) \right]$$
$$\ddot{\gamma}^k = - \langle \dot{\gamma}, \Gamma^k \dot{\gamma} \rangle$$

and can therefore be presented with respect to the Christoffel symbols.

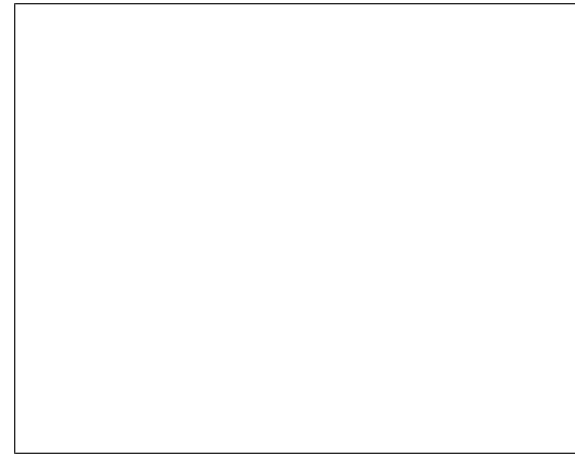
Example: Geodesics



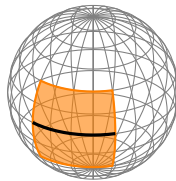
Parametrization



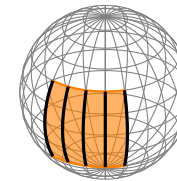
$\nabla_{\partial_1 x} \partial_1 x$



$\nabla_{\partial_2 x} \partial_2 x$



Equator



Meridians

Covariant Derivative along Curves

Given a curve $\gamma: (0, L) \rightarrow U$ and a vector field X along the curve $c = x \circ \gamma$, we would like to define

$$\frac{\nabla}{dt}X := \nabla_{\dot{c}}X$$

To this end let $Y = \sum_{i=1}^n y^i \partial_i x$ be a vector field on M that coincides along c with X . Further let $Z = \sum_{i=1}^n z^i \partial_i x$ be a vector field that coincides along c with \dot{c} . Then we have ($p = x(u) = c(\tau)$)

$$\nabla_Z Y(p) = \sum_{k=1}^n \left[\frac{d}{dt} \left(y^k \circ \gamma(t) \right) \Big|_{t=\tau} + \sum_{i,j=1}^n y^i(u) \Gamma_{ij}^k(u) z^j(u) \right] \partial_k x(u)$$

If we restrict this vector field to a vector field along the curve, it only depends on X and c , but not on the extension of Y and Z . Thus, $\frac{\nabla}{dt}$ is well defined.

Geodesic Equation in Terms of the Covariant Derivative

Given a geodesic $c: (0, 1) \rightarrow M$, we have for $\dot{c} = \sum_{i=1}^n \dot{\gamma}^i \partial_i x$

$$\begin{aligned}\frac{\nabla}{dt} \dot{c} &= \sum_{k=1}^n \left[\ddot{\gamma}^k + \sum_{i,j=1}^n \dot{\gamma}^i \Gamma_{ij}^k \dot{\gamma}^j \right] \partial_k x \\ &= \sum_{k=1}^n \left[\ddot{\gamma}^k + \langle \dot{\gamma}, \Gamma^k \dot{\gamma} \rangle \right] \partial_k x = 0\end{aligned}$$

The geodesic equation can therefore be written as

$$\frac{\nabla}{dt} \dot{c} = 0$$

Since $\frac{\nabla}{dt} \dot{c}$ measures how different a curve c is from a geodesic we can use it to define the **geodesic curvature** of a curve.

Geodesic curvature

Given a curve $c: (0, L) \rightarrow \mathbb{R}^2$ parametrized by arc-length ($\|\dot{c}\| \equiv 1$), the curvature $\kappa(t)$ at $c(t)$ can be computed via

$$\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3} = \det(\dot{c}(t), \ddot{c}(t))$$

Given a curve $c: (0, L) \rightarrow M$ in the 2D manifold M that is parametrized by arc-length, we can compute the **geodesic curvature** $\kappa_g(t)$ by replacing \ddot{c} with $\frac{\nabla}{dt}\dot{c}$ and obtain

$$\kappa_g(t) = \det\left(\dot{c}(t), \frac{\nabla}{dt}\dot{c}(t)\right)$$

The geodesic curvature is 0 for geodesics and can therefore be understood as an intrinsic reformulation of the classical curvature of curves.

Gauss Map

Given a 2D manifold $M \subset \mathbb{R}^3$, we call a smooth mapping

$$N: M \rightarrow \mathbb{S}^2 \qquad \forall p \in M: N(p) \perp T_p M$$

its **Gauss map**. For every 3D shape there exists such a mapping. (Why?)

If $x: U \rightarrow M$ is a coordinate mapping, we can always define a local Gauss map via

$$N: M \rightarrow \mathbb{S}^2$$

$$p \mapsto \frac{\partial_1 x(u) \times \partial_2 x(u)}{\|\partial_1 x(u) \times \partial_2 x(u)\|} \qquad \text{for } u = x^{-1}(p)$$

If $M = f^{-1}(c)$ is given implicitly via a function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, the Gauss map is given via $N(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$.

Shape Operator

Given a 2D manifold $M \subset \mathbb{R}^3$ together with its Gauss map $N: M \rightarrow \mathbb{S}^2$, we call its differential the **shape operator** or **Weingarten mapping** S

$$\begin{aligned} S_p: T_p M &\rightarrow T_{N(p)} \mathbb{S}^2 \\ v &\mapsto DN(p)[v] \end{aligned}$$

Since $T_{N(p)} \mathbb{S}^2 = N(p)^\perp = T_p M$, $S_p: T_p M \rightarrow T_p M$ is an endomorphism.

If we choose a basis of $T_p M$, we would obtain a 2×2 matrix, but this matrix would depend on the chosen basis. Nonetheless, the eigenvalues of these matrices would remain the same.

The goal is to show that S_p can be put in diagonal form and that both eigenvalues are real.

Self-Adjointness of the Shape Operator

We know (Linear Algebra) that self-adjoint endomorphisms are diagonalizable with real eigenvalues. Therefore, we have to prove that

$$\langle v_1, S_p(v_2) \rangle = \langle S_p(v_1), v_2 \rangle \quad \text{for all } v_1, v_2 \in T_p M$$

If v_1 and v_2 are co-linear this is obvious. If they are not co-linear, one can find a local coordinate map $x: U \rightarrow M$ with $x(0) = p$ and $v_i = \partial_i x(0)$.

Using $\langle N \circ x(u), \partial_i x(u) \rangle \equiv 0$ leads to

$$\begin{aligned} 0 &= \partial_1 \langle N \circ x(u), \partial_2 x(u) \rangle|_{u=0} = \langle S_p(v_1), v_2 \rangle + \langle N(p), \partial_{12} x(0) \rangle \\ 0 &= \partial_2 \langle N \circ x(u), \partial_1 x(u) \rangle|_{u=0} = \langle S_p(v_2), v_1 \rangle + \langle N(p), \partial_{21} x(0) \rangle \end{aligned}$$

which proves the self-adjointness of the shape operator.

Principal Curvatures

The two eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of S_p are called **principal curvatures** and corresponding eigenvectors $v_1(p)$ and $v_2(p)$ are called **principal curvature directions**.

Note that $\kappa_g(p)$ along the geodesic c_i corresponding to $v_i(p)$ is 0 and the curvature of this curve coincides with $\kappa_i(p)$. In that sense, we can think of the principal curvatures as natural generalizations of the planar curvature.

We can derive two other curvatures from the principal curvatures:

$$\begin{aligned} H(p) &:= \frac{\kappa_1(p) + \kappa_2(p)}{2} &= \frac{1}{2} \operatorname{tr}(\mathcal{M}) && \text{(mean curvature)} \\ K(p) &:= \kappa_1(p) \cdot \kappa_2(p) &= \det(\mathcal{M}) && \text{(Gauss curvature)} \end{aligned}$$

given a representing matrix \mathcal{M} of S_p .

Second Fundamental Form

Given the shape operator $S_p: T_pM \rightarrow T_pM$, we can define the **Second Fundamental Form**

$$\mathbb{I}: T_pM \times T_pM \rightarrow \mathbb{R} \quad (v_1, v_2) \mapsto \langle S_p v_1, v_2 \rangle$$

This means, we have

$$\partial_{ij}x = \sum_{k=1}^n \Gamma_{ij}^k \partial_k x - \mathbb{I}(\partial_i x, \partial_j x) \cdot N$$

and the second fundamental form can be computed via

$$\mathbb{I}(\partial_i x, \partial_j x) = - \langle \partial_{ij}x, N \rangle.$$

Shape Operator in Local Coordinates

Any coordinate map $x: U \rightarrow M$ provides for a base $\{\partial_1 x(u), \dots, \partial_n x(u)\}$ of $T_p M$ for $p = x(u)$. In this base, the shape operator S_p can be written as

$$S_p(\partial_j x(u)) = \sum_{i=1}^n \mathcal{M}_j^i \partial_i x(u)$$

This means, we have

$$-\mathbb{I}(\partial_j x, \partial_k x) = \langle S_p(\partial_j x), \partial_k x \rangle = \sum_{i=1}^n \langle \mathcal{M}_j^i \partial_i x, \partial_k x \rangle = \sum_{i=1}^n g_{ki} \mathcal{M}_j^i$$

In other words the representating matrix \mathcal{M} of S_p satisfies the **Weingarten equations**

$$\mathcal{M} = -g^{-1} \cdot \mathbb{I}$$