Analysis of Three-Dimensional Shapes
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Weekly Exercises 1

Room: 02.09.023
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Mathematics: Calculus recap and Manifolds

Recap the definition of *partial derivative* if you are not familiar with it anymore. Quick introduction of notation: For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ the partial derivative of the $j-th$ component of f by the i-th variable can be written as

1. $\partial_i f^j$ with $i \in \{1, ..., n\}, j \in \{1, ..., m\}$
2. $\frac{\partial f^j}{\partial x_i}$ describing the same thing but assuming that the variable are given names as is normally case (e.g. $(x, y, z) \mapsto (x, y + z)$)

The notation is a matter of taste but some are less confusion depending on the situation.

The *differential* is the best linear approximation of a function. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$ it can be represented by its Jacobi matrix:

$$Df = \begin{pmatrix} \partial_1 f^1 & \dots & \partial_n f^1 \\ \vdots & & \vdots \\ \partial_1 f^m & \dots & \partial_n f^m \end{pmatrix}$$

or (if taking partial derivatives is not trivial)

$$Df(x)[h] \doteq f(x + h) - f(x)$$

In this case the equality holds only for linear terms in h.

Exercise 1 (2 points). 1. Let f be

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases} \end{aligned}$$

Calculate the partial derivatives $\partial_1 f$ and $\partial_2 f$. What happens at $\partial_1 f(0, 0)$?

2. Consider $g: \mathbb{R}^2 \to \mathbb{R}$ and $f: \mathbb{R}^2 \to \mathbb{R}$ differentiable with

$$g(x_1, x_2) = f(x_1^2, x_1 + x_2)$$

   Calculate $\frac{\partial g}{\partial x_1}$ (in relation to f).

Exercise 2 (2 points).

1. Calculate the differential of

$$\begin{aligned} f_1 : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (x(1-y), xyz) \end{aligned}$$

2. Calculate the differential of

$$\begin{aligned} f_2 : \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u, v) &\mapsto (u^2 + v^2, u - v, 4v^4) \end{aligned}$$

Exercise 3 (3 points).

1. Consider the function

$$\begin{aligned} g_1 : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (t^3, t^2) \end{aligned} \tag{1}$$

   Calculate the differential of g_1.

2. Reason whether g_1 is an explicit representation of a manifold.
   Tip: A vector is not full-rank if its 0.

3. Consider the next function

$$\begin{aligned} g_2 : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (t, t^3, t^2) \end{aligned} \tag{2}$$

   Reason whether g_2 is an explicit representation of a manifold.
   Tip: Ex. 5 can help imagining g_1, g_2.

Exercise 4 (3 points). Consider the set $O(n)$ including orthogonal matrices in $\mathbb{R}^{n \times n}$

$$O(n) = \{A \in \mathbb{R}^{n \times n} \mid AA^\top = Id\}$$

and the following map

$$\begin{aligned} \varphi : \mathbb{R}^{n \times n} &\to Sym(n) \\ A &\mapsto AA^\top \end{aligned}$$

1. Calculate the differential of φ.

2. $O(n)$ can be described by the implicit formulation $O(n) = \varphi^{-1}(Id)$. Proof that $O(n)$ is a manifold by showing that the differential of φ is of full rank.

3. What is the dimension of $O(n)$? Explain your answer.

Programming: Working with Matlab

You can use MATLAB in the computer labs or install it on your own computer (licences are free for TUM students, see matlab.rbg.tum.de). If you have never used MATLAB before, going through a tutorial before doing the exercises will probably help (there is an interactive MATLAB Academy).

Exercise 5 (1 point)**.** Consider the functions (1), (2) from Ex. 3. Plot both functions for $t \in [-2, 2]$ in two subplots. Include a figure of your result in your submission. Tip: Looking up `subplot`, `plot`, `plot3` and function handles (if you feel fancy) might be helpful.

Exercise 6 (3 points)**.** The unit sphere can be described as

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$$

Consider the three following coordinate maps: ($C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$)

$$\begin{aligned} x_1 &: C \to \mathbb{R}^3 \\ &\quad (u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2}) \quad (3) \\ x_2 &: C \to \mathbb{R}^3 \\ &\quad (u, v) \mapsto (u, v, -\sqrt{1 - u^2 - v^2}) \quad (4) \\ x_3 &:]-10, 10[\times]-10, 10[\to \mathbb{R}^3 \\ &\quad (u, v) \mapsto \frac{1}{u^2 + v^2 + 1}(2u, 2v, u^2 + v^2 - 1) \quad (5) \\ x_4 &:]0, 1[\times]0, 2\pi[\to \mathbb{R}^3 \\ &\quad (h, \theta) \mapsto (\sin(h\pi)\cos(\theta), \sin(h\pi)\sin(\theta), \cos(h\pi)) \quad (6) \end{aligned}$$

The height function h on S is defined as follows:

$$\begin{aligned} h &: S \to \mathbb{R} \\ &\quad (x, y, z) \mapsto z \quad (7) \end{aligned}$$

1. Make a figure plotting the images of x_1, x_2, x_3 as surfaces with h as a function on the surface.

2. Plot $(x_1^{-1} \circ h)$, $(x_2^{-1} \circ h)$ and $(x_3^{-1} \circ h)$ as 2D images.

3. Which pairs of coordinates maps together properly define the unit sphere as a manifold?

Tip: Look up the Matlab functions `ndgrid`, `surf` (set z-coordinate to NaN if you do not want some values to be plotted, surf(..., 'EdgeAlpha', 0) will make the edges invisible), `imagesc` and logical indexing.