Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
SS 2017

Tao Wu
Thomas Möllenhoff
Emanuel Laude

Computer Vision Group
Department of Informatics
TU München

Convex Set

Tao Wu
Thomas Möllenhoff
Emanuel Laude

Convex Optimization

Assume

- $\mathbb{E}$ is a Euclidean space (finite dimensional vector space), equipped with the inner product $\langle \cdot, \cdot \rangle$, e.g. $\langle u, v \rangle = u \cdot v$.
- C is a closed convex subset in $\mathbb{E}$.
- J is a convex objective function.

Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

- What is a convex set?
- What is a convex function?

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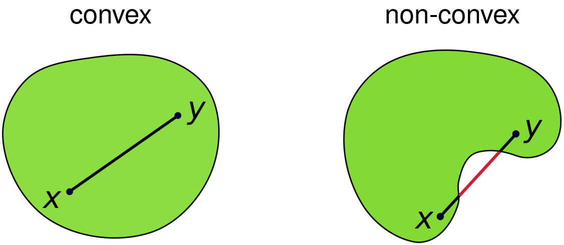
Thomas Möllenhoff
Emanuel Laude

Convex set

Definition

A set C is said to be **convex** if

$$\alpha u + (1-\alpha)v \in C, \quad \forall u, v \in C, \; \forall \alpha \in [0,1].$$



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Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C$, $\exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\operatorname{cl} C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\mathrm{int}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a convex set $C \subset \mathbb{E}$ is

$$\mathrm{ri}\, C = \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\}.$$

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Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$
- Closure: $\mathrm{cl}\, C$
- Interior: $\mathrm{int}\, C$

– The union of convex sets is not convex in general.

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– The union of convex sets is not convex in general.

– *Polyhedral sets* are always convex; *cones* are not necessarily convex.

Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$. C is a **convex cone** if C is a cone and is convex as well.

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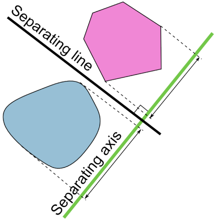
Separation of convex sets



Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets in $\mathbb{E}$ s.t. $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then there exists a hyperplane separating C_1 and C_2, i.e. $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.

$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Proof: on board.

Separation of convex sets

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Proof: on board.

Remarks

1. The proof works in any real Hilbert space.
2. Corollary: In a real Hilbert space, any (strongly) closed convex subset C is weakly closed.

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Thomas Möllenhoff
Emanuel Laude

Convex Function

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Convex functions

- An **extended real-valued function** J maps from $\mathbb{E}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.
- The **domain** of $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{dom} J = \{u \in \mathbb{E} : J(u) < \infty\}.$$

- The function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is **proper** if $\operatorname{dom} J \neq \emptyset$.

Definition

We say $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a **convex function** if

1. $\operatorname{dom} J$ is a convex set.
2. For all $u, v \in \operatorname{dom} J$ and $\alpha \in [0, 1]$ it holds that

$$J(\alpha u + (1 - \alpha)v) \leq \alpha J(u) + (1 - \alpha)J(v).$$

We say J is **strictly convex** if the above inequality is strict for all $\alpha \in (0, 1)$ and $u \neq v$.

Examples

- $J_{data}(u) = \|u - f\|_q^q$ where $q \geq 1$ and $\|\cdot\|_q$ is the ℓ^q-norm.
- $J_{regu}(u) = \|Ku\|_q^q$ where K is linear transform.
- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$ where $\alpha > 0$.

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- **Indicator function**:

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise,} \end{cases}$$

  where C is a convex subset of $\mathbb{E}$.

- Alternative formulation of constrained optimization:

$$\min J(u) \text{ over } u \in C. \quad \Leftrightarrow \quad \min J(u) + \delta_C(u) \text{ over } u \in \mathbb{E}.$$

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Basic facts

(As exercises)

- Any norm (over a normed vector space) is a convex function.
- J is a convex function and K is a linear transform $\Rightarrow J(K\cdot)$ is convex function.
- (Jensen's inequality) $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex iff

$$J(\sum_{i=1}^{n} \alpha_i u^i) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

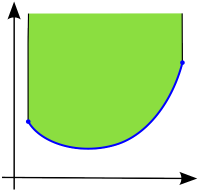
whenever $\{u^i\}_{i=1}^n \subset \mathbb{E}$, $\{\alpha_i\}_{i=1}^n \subset [0,1]$, $\sum_{i=1}^n \alpha_i = 1$.

Epigraph

Definition

The **epigraph** of a proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{epi} J = \{(u, \alpha) \in \mathbb{E} \times \mathbb{R} : J(u) \leq \alpha\}.$$



epigraph

Theorem

A proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex (resp. strictly convex) iff epi J is a convex (resp. strictly convex) set.

Proof: as exercise.

Tao Wu
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Lipschitz continuity

Definition

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ with $\operatorname{ri}\operatorname{dom} J \neq \emptyset$. We say J is **locally Lipschitz** at $u \in \operatorname{ri}\operatorname{dom} J$ with modulus $L_u > 0$ if there exists $\epsilon > 0$ s.t.

$$|J(u^1) - J(u^2)| \leq L_u \|u^1 - u^2\| \quad \forall u^1, u^2 \in B_\epsilon(u) \cap \operatorname{ri}\operatorname{dom} J.$$

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Theorem

A proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \operatorname{ri}\operatorname{dom} J$.

Proof: on board.

Global vs. Local minimizers

Recall the optimization of $J : \mathbb{E} \to \overline{\mathbb{R}}$:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Definition

1. $u^* \in \mathbb{E}$ is a **global minimizer** if $J(u^*) \le J(u)$ for all $u \in \mathbb{E}$.
2. u^* is a **local minimizer** if $\exists \epsilon > 0$ s.t. $J(u^*) \le J(u)$ for all $u \in B_\epsilon(u^*)$.
3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \le J(u)$ is replaced by strict inequality $J(u^*) < J(u)$.

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3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by strict inequality $J(u^*) < J(u)$.

Theorem

For any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, if $u^* \in \operatorname{dom} J$ is a local minimizer of J, then it is also a global minimizer.

Proof: on board.

Existence of Minimizer

Tao Wu
Thomas Möllenhoff
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Does a minimizer always exist?

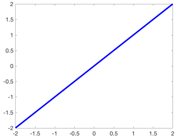
- Consider

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E},$$

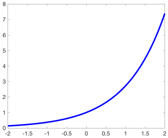
where $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a proper, convex function.

- Some counterexamples for $J : \mathbb{R} \to \overline{\mathbb{R}}$:

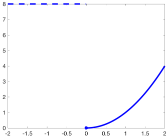
u



$\exp u$



$u^2 + \delta\{u > 0\}$



- We shall formalize our observations and derive sufficient conditions for existence.

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Sufficient conditions for existence

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Thomas Möllenhoff
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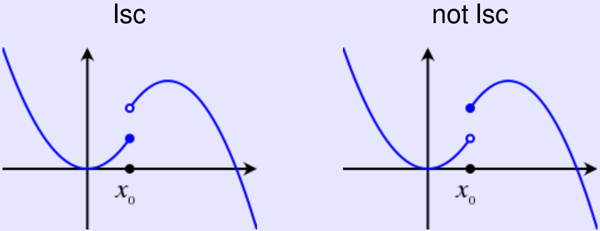
Definition

1. J is **bounded from below** if $J(\cdot) \geq C$ for some $C \in \mathbb{R}$.
2. J is **coercive** if $J(u) \to \infty$ whenever $\|u\| \to \infty$.
   - Proposition: J is coercive if dom J is bounded.
3. J is **lower-semicontinuous** (lsc) if

$$J(u^*) \leq \liminf_{u \to u^*} J(u).$$



   - Proposition: J is lsc iff epi J is closed.

Sufficient conditions for existence

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Theorem

Any proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc, has a (global) minimizer.

Proof: on board.

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Remarks for infinite dimensions

1. Weak compactness in reflexive Banach (e.g. Hilbert) sp.
2. J is convex and strongly lsc $\Rightarrow$ J is weakly lsc

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Uniqueness

- Recall that a function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is strictly convex if

$$J(\alpha u + (1-\alpha)v) < \alpha J(u) + (1-\alpha)J(v),$$

for all $u, v \in \operatorname{dom} J,\ u \neq v,\ \alpha \in (0,1)$.

Theorem

The minimizer of a strictly convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is unique.

Proof: on board.

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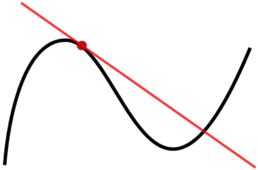
Subdifferential

Differential



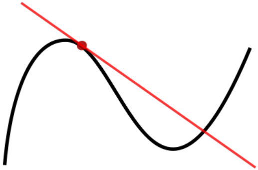
Definition

$J : \mathbb{E} \to \overline{\mathbb{R}}$ is called (Fréchet) **differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ and $\nabla J(u) \in \mathbb{E}$ is the (Fréchet) **differential** of J at u if

$$\lim_{h \to 0,\, h \in (\operatorname{dom} J) - u} \frac{J(u+h) - J(u) - \langle \nabla J(u), h \rangle}{\|h\|} = 0.$$

J is said **continuously differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $\operatorname{dom} J \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Differential



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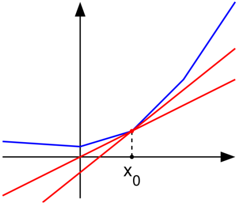
J is said **continuously differentiable** at $u \in \operatorname{int}\operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $\operatorname{dom} J \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Remark

In functional analysis, $\nabla J(u)$ is treated as a *dual* object in $\mathbb{E}^*$, and $\langle \nabla J(u), h \rangle_{\mathbb{E}^*, \mathbb{E}}$ as *duality pairing*.

Subdifferential

Now we generalize differentiability from differentiable functions to nonsmooth (convex) functions.



Definition

The **subdifferential** of a convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ at $u \in \operatorname{dom} J$ is defined by

$$\partial J(u) = \{\xi \in \mathbb{E} : J(v) \geq J(u) + \langle \xi, v - u \rangle\} .$$

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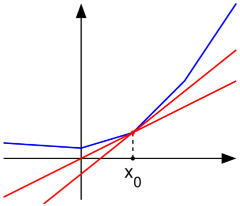

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Geometric interpretation

$\xi \in \partial J(u)$ iff $(\xi, -1)$ is a normal vector for the supporting hyperplane of $\operatorname{epi} J$ at $(u, J(u))$.

Tao Wu
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Examples and basic facts

Basic facts

1. $\partial J(\cdot)$ is a **set-valued map**.
2. If J is cont. differentiable at u, then $\partial J(u) = \{\nabla J(u)\}$.

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Examples and basic facts

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Examples (as exercises)

1. $u \in \mathbb{R}^n \mapsto \|u\|_1$.
2. $u \in \mathbb{R}^n \mapsto \|u\|_\infty$.
3. $X \in \mathbb{R}^{n\times n} \mapsto \|X\|_{1,2} = \sum_i \left(\sum_j |X_{i,j}|^2\right)^{1/2}$.
4. $X \in \mathbb{R}^{n\times n} \mapsto \|X\|_{nuc} = \sum_i \sigma_i(X)$ (sum of singular values).
5. Given a closed convex subset $C \subset \mathbb{E}$ and $u \in C$,

$$\partial\delta_C(u) = N_C(u) = \{\xi \in \mathbb{E} : \langle \xi, v - u\rangle \leq 0\ \forall v \in C\},$$

known as the **normal cone** of C at u.

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Subdifferential calculus

Theorem (chain rule under linear transform)

Let $\widetilde{J}(\cdot) = J(K\cdot)$ with some convex function J and linear transform K. Then

$$\partial\widetilde{J}(u) = K^\top \partial J(Ku)$$

for any $u \in \operatorname{ri}\operatorname{dom} J$.

Example: $J(u) = \|Ku\|_1 \Rightarrow \partial J(u) = K^\top \partial\|\cdot\|_1(Ku)$.

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Theorem (summation rule)

If $J(\cdot) = J_1(\cdot) + J_2(\cdot)$ for some convex functions J_1 and J_2, then

$$\partial \widetilde{J}(u) = \partial J_1(u) + \partial J_2(u)$$

for any $u \in \operatorname{ri}\operatorname{dom} J_1 \cap \operatorname{ri}\operatorname{dom} J_2$.

Warning: not true if J_1 or J_2 is non-convex, e.g. $0 = |\cdot| + (-|\cdot|)$.

Properties of subdifferential map

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then ∂J is a **monotone operator**, i.e. $\forall u^1, u^2 \in \operatorname{dom} J,\ \xi^1 \in \partial J(u^1),\ \xi^2 \in \partial J(u^2)$:

$$\left\langle \xi^1 - \xi^2, u^1 - u^2 \right\rangle \geq 0.$$

Proof: on board.

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Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \operatorname{int} \operatorname{dom} J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then the set-valued map $\partial J(\cdot)$ is **closed**, i.e. $\xi^* \in \partial J(u^*)$ whenever

$$\exists (u^k, \xi^k) \to (u^*, \xi^*) \in (\operatorname{ri} \operatorname{dom} J) \times \mathbb{E} \ \text{ s.t. } \xi^k \in \partial J(u^k)\ \forall k.$$

Proof: on board.

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Optimality condition

Theorem

Given any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is

$$0 \in \partial J(u^*).$$

Proof: on board.

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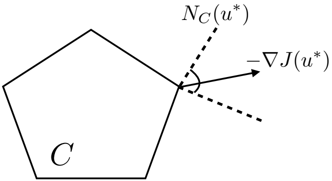
Proof: on board.

Constrained optimization as special case

If $\widetilde{J}(u) = J(u) + \delta_C(u)$ with convex function $J : \mathbb{E} \to \mathbb{R}$ and closed convex subset $C \in \mathbb{E}$, then $0 \in \partial \widetilde{J}(u^*) \rightsquigarrow$

$$0 \in \partial J(u^*) + N_C(u^*).$$



Tao Wu
Thomas Möllenhoff
Emanuel Laude

Optimality condition

Theorem

Given any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is

$$0 \in \partial J(u^*).$$

Proof: on board.

Constrained optimization as special case

If $\widetilde{J}(u) = J(u) + \delta_C(u)$ with convex function $J : \mathbb{E} \to \mathbb{R}$ and closed convex subset $C \in \mathbb{E}$, then $0 \in \partial \widetilde{J}(u^*) \rightsquigarrow$

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Remark

$0 \in \partial J(u^*) + N_C(u^*)$ is a *geometric* optimality condition. Further characterization replies on the algebraic representation of $N_C(u^*)$, e.g. the Karush-Kuhn-Tucker (KKT) conditions under certain constraint qualifications.

Tao Wu
Thomas Möllenhoff
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Convex Conjugate

Tao Wu
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Legendre-Fenchel transform

Definition

Given a convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the **convex conjugate** of J is defined by

$$J^*(p) = \sup_{u \in \mathbb{E}} \langle u, p \rangle - J(u).$$

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Examples (as exercises)

1. $J(\cdot) = \langle w, \cdot \rangle \;\Rightarrow\; J^*(\cdot) = \delta\{\cdot = w\}$. $(w \in \mathbb{E})$
2. $J(\cdot) = \frac{1}{q}\|\cdot\|_q^q \;\Rightarrow\; J^*(\cdot) = \frac{1}{q'}\|\cdot\|_{q'}^{q'}$. $(0 < q < 1, \frac{1}{q} + \frac{1}{q'} = 1)$
3. $J(\cdot) = \|\cdot\| \;\Rightarrow\; J^*(\cdot) = \delta\{\|\cdot\|_* \leq 1\}$. ($\|\cdot\|_*$ is the *dual norm* of $\|\cdot\|$, i.e. $\|v\|_* = \sup_{\|u\| \leq 1} \langle u, v \rangle$)
4. $J(\cdot) = \delta\{\|\cdot\|_* \leq 1\} \;\Rightarrow\; J^*(\cdot) = \|\cdot\|$.

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4. $J(\cdot) = \delta\{\|\cdot\|_* \le 1\} \;\Rightarrow\; J^*(\cdot) = \|\cdot\|$.

Basic facts

- Scalar multiplication: $\widetilde{J}(\cdot) = \alpha J(\cdot) \;\Rightarrow\; \widetilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$.
- Translation: $\widetilde{J}(\cdot) = J(\cdot - z) \;\Rightarrow\; \widetilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z \rangle$.

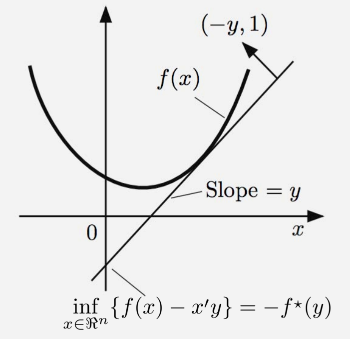
Geometric interpretation

Convex conjugation maps:

(non-vertical) supporting hyperplane of the epigraph

to:

intersection with the vertical axis.



Courtesy of Bertsekas

Fenchel-Young inequality

Theorem (Fenchel-Young inequality)

For all $u \in \operatorname{dom} J,\ p \in \operatorname{dom} J^*$, we have

$$J(u) + J^*(p) \geq \langle u, p \rangle \, .$$

The equality holds iff $p \in \partial J(u)$.

<u>Proof</u>: $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate; $p \in \partial J(u)$ is the sufficient and necessary condition for: $u = \arg\min_{v \in \mathbb{E}} J(v) - \langle v, p \rangle$.

Tao Wu
Thomas Möllenhoff
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Biconjugate

Theorem

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ and $J^{**} = (J^*)^*$ is the **biconjugate** of J.

In general:

1. $J^{**}(\cdot) \leq J(\cdot)$.
2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.
4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof: on board.

Tao Wu
Thomas Möllenhoff
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Regularity of J and J^*

Definition

$J : \mathbb{E} \to \mathbb{R}$ is said to be μ-**strongly convex** if $\exists \mu > 0$ s.t. $J(\cdot) - \frac{\mu}{2}\| \cdot \|^2$ is convex.

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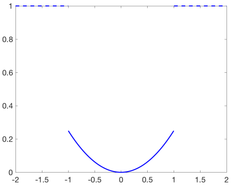
$J : \mathbb{E} \to \mathbb{R}$ is said to be μ-**strongly convex** if $\exists \mu > 0$ s.t. $J(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex.

Theorem

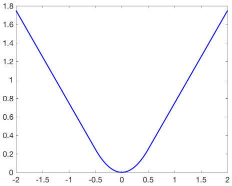
Assume that $J : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ-strongly convex iff J^* has $\frac{1}{\mu}$-Lipschitz gradient.

Proof: on board.

$J =$ truncated quadratic



$J^* =$ Huber function



Duality Theory

Fenchel-Rockafellar duality

- Consider
$$\min_{u\in\mathbb{R}^n} F(Ku) + G(u),$$
where $K \in \mathbb{R}^{m\times n}$, and F, G are proper, convex, and lsc.

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$$\min_{u \in \mathbb{R}^n} F(Ku) + G(u),$$

where $K \in \mathbb{R}^{m \times n}$, and F, G are proper, convex, and lsc.

- The **weak duality** always holds:

$$\begin{aligned}
\mathcal{P}^* &= \min_{u \in \mathbb{R}^n} F(Ku) + G(u) = \min_{u} \sup_{p} \langle p, Ku \rangle - F^*(p) + G(u) \\
&\geq \sup_{p} \inf_{u} \left\langle K^\top p, u \right\rangle + G(u) - F^*(p) \\
&= \sup_{p} -G^*(-K^\top p) - F^*(p) = \mathcal{D}^*.
\end{aligned}$$

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$$\begin{aligned}\mathcal{P}^* &= \min_{u \in \mathbb{R}^n} F(Ku) + G(u) = \min_u \sup_p \langle p, Ku \rangle - F^*(p) + G(u) \\ &\geq \sup_p \inf_u \left\langle K^\top p, u \right\rangle + G(u) - F^*(p) \\ &= \sup_p -G^*(-K^\top p) - F^*(p) = \mathcal{D}^*.\end{aligned}$$

- Define the **duality gap**:

$$\mathcal{G}(u, p) = F(Ku) + G(u) + G^*(-K^\top p) + F^*(p).$$

Note that $\mathcal{G}(u, p) = 0$ is an optimality criterion.

Fenchel-Rockafellar duality

- $\mathcal{G}(u^*, p^*) = 0 \Leftrightarrow \mathcal{P}^* = \mathcal{D}^* \Leftrightarrow (u^*, p^*)$ solves the **saddle point problem** with $\mathcal{L}(u,p) := \langle p, Ku \rangle - F^*(p) + G(u)$:

$$\mathcal{L}(u^*, p) \leq \mathcal{L}(u^*, p^*) \leq \mathcal{L}(u, p^*) \quad \forall (u,p).$$

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Theorem (Fenchel-Rockafellar duality)

Assume $\exists \bar{u} \in \operatorname{dom} G$ s.t. F is continuous at $K\bar{u}$. Then the **strong duality** holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal solution pair iff

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof: on board.

Tao Wu
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Application: Total-variation image restoration

- Primal problem (let $\alpha > 0,\ q \in [1, \infty],\ \Omega \subset \mathbb{R}^d,\ f \in \mathbb{R}^\Omega$):

$$\min_{u \in \mathbb{R}^\Omega} \alpha \|\nabla u\|_{1,q} + \frac{1}{2}\|u - f\|^2.$$

Here $\|p\|_{1,q} = \sum_{j \in \Omega} |p_j|_{\ell^q}$ for each $p \in (\mathbb{R}^\Omega)^d$.

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- Apply the Fenchel-Rockafellar duality with

$$F(\cdot) = \alpha\|\cdot\|_{1,q}, \quad K = \nabla, \quad G(\cdot) = \frac{1}{2}\|\cdot - f\|^2.$$

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- Saddle point problem ($1/q + 1/q' = 1$):

$$\max_p \min_u \langle p, \nabla u \rangle - \delta\{\|p\|_{\infty,q'} \leq \alpha\} + \frac{1}{2}\|u - f\|^2.$$

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- Dual problem:

$$\min_p \frac{1}{2}\|\nabla^\top p - f\|^2 + \delta\{\|p\|_{\infty,q'} \leq \alpha\}.$$

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Proximal Operator

Proximal operator

Definition

Given a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$, we define the **proximal operator** of J by

$$\operatorname{prox}_{\tau J}(v) = \arg\min_{u} J(u) + \frac{1}{2\tau}\|u - v\|^2.$$

<u>Note</u>: The above minimization always has a unique minimizer.

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Proximal operator in algorithms

Let us solve the convex optimization:

$$\min_{u} F(u) + G(u),$$

where G is cont'ly differentiable but F is non-differentiable.
The **proximal gradient** iteration appears as:

$$u^{k+1} = \operatorname{prox}_{\tau F}(u^k - \tau \nabla G(u^k)).$$

Tao Wu
Thomas Möllenhoff
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Observations

1. By checking the optimality condition,

$$u = \text{prox}_{\tau J}(v) \;\Leftrightarrow\; 0 \in \tau \partial J(u) + u - v \;\Leftrightarrow\; u = (I + \tau \partial J)^{-1}(v).$$

Thus, $\text{prox}_{\tau J} = (I + \tau \partial J)^{-1}$, known as the **resolvent** of ∂J.

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2. u^* is a **fixed point** of $\text{prox}_{\tau J}$, i.e. $u^* = \text{prox}_{\tau J}(u^*)$,
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3. Derivation of proximal gradient algorithm:

$$\begin{aligned}
& u^* \in \arg\min_u F(u) + G(u) \\
\Leftrightarrow\;\; & 0 \in \partial F(u^*) + \nabla G(u^*) \\
\Leftrightarrow\;\; & u^* + \tau\partial F(u^*) \ni u^* - \tau\nabla G(u^*) \\
\Leftrightarrow\;\; & u^* = (I + \tau\partial F)^{-1}(u^* - \tau\nabla G(u^*)) \\
\rightsquigarrow\;\; & u^{k+1} = \operatorname{prox}_{\tau F}(u^k - \tau\nabla G(u^k)).
\end{aligned}$$

Logistic regression (programming exercise)

- MNIST dataset[1] - handwritten digit recognition.
- Train classifier on training set; Evaluate on test set.
- Conv. neural network: 0.23%; **Logistic regression**: 10%.

[1] http://yann.lecun.com/exdb/mnist/

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- $K = 10$ classes; grayscale images of pixels $M = 28 \times 28$.
- $N = 60000$ training images $X \in \mathbb{R}^{N \times M}$, with ground-truth labels $Y \in \{1, ..., K\}^N$.
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- Train **linear classifier** $W \in \mathbb{R}^{K \times M}$ on raw pixel data.
- **Softmax** function:

$$\mathcal{P}(Y_n = k | X_{n,\cdot}, W) = \frac{\exp(\langle W_{k,\cdot}, X_{n,\cdot} \rangle + b^k)}{\sum_{k'=1}^{K} \exp(\langle W_{k',\cdot}, X_{n,\cdot} \rangle + b_{k'})}.$$

- Minimize negative log-likelihood + regularizer:

$$\min_{W \in \mathbb{R}^{K \times M},\, b \in \mathbb{R}^K} R(W) - \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbf{1}\{Y_n = k\} \log \mathcal{P}(Y_n = k | X_{n,\cdot}, W).$$

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Examples

1. Indicator function: Let C be a closed, convex subset $\Rightarrow$

$$\text{prox}_{\tau \delta_C}(\cdot) = \text{proj}_C(\cdot).$$

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$$\begin{aligned}\mathrm{prox}_{\tau\widehat{J}}(\bar{u}) &= \arg\min_u \frac{1}{2\tau}\|u - \bar{u}\|^2 + \langle \nabla J(\bar{u}), u - \bar{u} \rangle \\ &= \bar{u} - \tau\nabla J(\bar{u}). \quad \text{(gradient descent step)}\end{aligned}$$

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3. Quadratic approximation:
   $\widetilde{J}(\cdot) := J(\bar{u}) + \langle \nabla J(\bar{u}), \cdot - \bar{u} \rangle + \frac{1}{2}\left\langle \nabla^2 J(\bar{u})(\cdot - \bar{u}), \cdot - \bar{u} \right\rangle \Rightarrow$

$$\begin{aligned}\operatorname{prox}_{\tau\widetilde{J}}(\bar{u}) &= \arg\min_u \frac{1}{2\tau}\|u - \bar{u}\|^2 + \langle \nabla J(\bar{u}), u - \bar{u} \rangle \\ &\quad + \frac{1}{2}\left\langle \nabla^2 J(\bar{u})(u - \bar{u}), u - \bar{u} \right\rangle \\ &= \bar{u} - (\nabla^2 J(\bar{u}) + \frac{1}{\tau} I)^{-1} \nabla J(\bar{u}). \quad \text{(damped Newton step)}\end{aligned}$$

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Moreau identity

Theorem (Moreau identity)

Let $\tau > 0$ and $J : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then the following identity holds:

$$\mathrm{id}(\cdot) = \mathrm{prox}_{\tau J}(\cdot) + \tau\, \mathrm{prox}_{\frac{1}{\tau} J^*}(\cdot/\tau).$$

In particular, $\tau = 1 \Rightarrow \mathrm{id}(\cdot) = \mathrm{prox}_{J}(\cdot) + \mathrm{prox}_{J^*}(\cdot)$.

<u>Proof</u>:

$$\begin{aligned} & v = \tau\, \mathrm{prox}_{\frac{1}{\tau} J^*}(u/\tau) \\ \Leftrightarrow\ & \left(I + \frac{1}{\tau}\partial J^*\right)^{-1}(u/\tau) = v/\tau \\ \Leftrightarrow\ & \partial J^*(v/\tau) \ni u - v \\ \Leftrightarrow\ & v/\tau \in \partial J(u - v) \\ \Leftrightarrow\ & u - v = (I + \tau\partial J)^{-1}(u) = \mathrm{prox}_{\tau J}(u). \end{aligned}$$

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Remark

The Moreau identity suggests that if one of $\mathrm{prox}_J(\cdot)$ and $\mathrm{prox}_{J^*}(\cdot)$ is computable, so is the other.

Infimal convolution

Definition

Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. The **infimal convolution** (or inf convolution) of F and G is defined by

$$(F \Box G)(\cdot) = \inf_{u \in \mathbb{E}} \{F(\cdot - u) + G(u)\},$$

with $\text{dom}(F \Box G) = \text{dom}\, F + \text{dom}\, G$.

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Let $F, G : \mathbb{E} \to \overline{\mathbb{R}}$ be proper, convex, and lsc. Then

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<u>Proof</u>: on board.

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$$(F \square G)^*(\cdot) = F^*(\cdot) + G^*(\cdot).$$

Proof: on board.

Remark

Analogy between inf convolution (under convex conjugation) and convolution (under Fourier transform): $\widehat{F * G}(\cdot) = \widehat{F}(\cdot)\widehat{G}(\cdot)$.

Tao Wu
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Moreau envelope

Definition

The **Moreau envelope** of a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is defined for each $u \in \mathbb{E}$ by

$$\begin{aligned}\mathcal{M}_{\tau J}(u) &:= \left(J \,\square \left(\frac{1}{2\tau} \| \cdot \|^2 \right) \right) (u) \\ &= \inf_{v \in \mathbb{E}} \left\{ J(v) + \frac{1}{2\tau} \| v - u \|^2 \right\} \\ &= J(\text{prox}_{\tau J}(u)) + \frac{1}{2\tau} \| \text{prox}_{\tau J}(u) - u \|^2.\end{aligned}$$

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Definition

The **Moreau envelope** of a proper, convex, lsc function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is defined for each $u \in \mathbb{E}$ by

$$\begin{aligned}\mathcal{M}_{\tau J}(u) &:= \left(J \,\square \left(\frac{1}{2\tau}\|\cdot\|^2\right)\right)(u) \\ &= \inf_{v \in \mathbb{E}} \left\{ J(v) + \frac{1}{2\tau}\|v - u\|^2 \right\} \\ &= J(\mathrm{prox}_{\tau J}(u)) + \frac{1}{2\tau}\|\mathrm{prox}_{\tau J}(u) - u\|^2.\end{aligned}$$

Example

$J : u \in \mathbb{R} \mapsto |u| \Rightarrow \mathcal{M}_{\tau J}$ is Huber function:

$$\mathcal{M}_{\tau J}(u) = \begin{cases} u^2/(2\tau) & \text{if } |u| \le \tau, \\ |u| - \tau/2 & \text{if } |u| > \tau. \end{cases}$$

Observation: $\mathcal{M}_{\tau J}$ does smoothing/regularization on J.

Properties

- Recall the theorem: $(F \square G)^*(\cdot) = F^*(\cdot) + G^*(\cdot) \Rightarrow$

$$(\mathcal{M}_{\tau J})^* = J^* + (\frac{1}{2\tau}\|\cdot\|^2)^* = J^* + \frac{\tau}{2}\|\cdot\|^2.$$

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- $\nabla\mathcal{M}_{\tau J}$ can be calculated as:

$$\begin{aligned}
p = \nabla\mathcal{M}_{\tau J}(u) \;&\Leftrightarrow\; u \in \partial(\mathcal{M}_{\tau J})^*(p) = \partial J^*(p) + \tau p \\
&\Leftrightarrow\; u - \tau p \in \partial J^*(p) \;\Leftrightarrow\; \partial J(u - \tau p) \ni p \\
&\Leftrightarrow\; \tau\partial J(u - \tau p) \ni \tau p \;\Leftrightarrow\; (I + \tau\partial J)(u - \tau p) \ni u \\
&\Leftrightarrow\; u - \tau p = (I + \tau\partial J)^{-1}u = \mathrm{prox}_{\tau J}(u) \\
&\Leftrightarrow\; \nabla\mathcal{M}_{\tau J}(u) = \frac{1}{\tau}(u - \mathrm{prox}_{\tau J}(u)).
\end{aligned}$$