Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
SS 2017

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Convex Set

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Convex Optimization

Assume

- $\mathbb{E}$ is a Euclidean space (finite dimensional vector space), equipped with the inner product $\langle \cdot, \cdot \rangle$, e.g. $\langle u, v \rangle = u \cdot v$.
- C is a closed convex subset in $\mathbb{E}$.
- J is a convex objective function.

Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

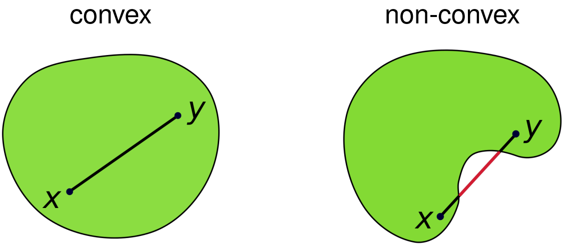
- What is a convex set?
- What is a convex function?

Convex set

Definition

A set C is said to be **convex** if

$$\alpha u + (1 - \alpha) v \in C, \quad \forall u, v \in C, \ \forall \alpha \in [0, 1].$$



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Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C$, $\exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\operatorname{cl} C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a <u>convex</u> set $C \subset \mathbb{E}$ is

$$\operatorname{ri} C = \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\}.$$

Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$
- Closure: $\operatorname{cl} C$
- Interior: $\operatorname{int} C$

– The union of convex sets is not convex in general.

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Basic properties

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- Interior: $\mathrm{int}\, C$

– The union of convex sets is not convex in general.

– *Polyhedral sets* are always convex; *cones* are not necessarily convex.

Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$. C is a **convex cone** if C is a cone and is convex as well.

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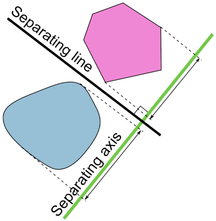
Separation of convex sets



Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets in $\mathbb{E}$ such that $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then there exists a hyperplane separating C_1 and C_2, i.e. $\exists v \in \mathbb{E},\ \alpha \in \mathbb{R}$ such that

$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Proof: on board.

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Separation of convex sets

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Proof: on board.

Remarks

1. The proof works in any real Hilbert space.
2. Corollary: In a real Hilbert space, any (strongly) closed convex subset C is weakly closed.

Convex Function

Convex functions

- An **extended real-valued function** J maps from $\mathbb{E}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.
- The **domain** of $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{dom} J = \{u \in \mathbb{E} : J(u) < \infty\}.$$

- The function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is **proper** if $\operatorname{dom} J \neq \emptyset$.

Definition

We say $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a **convex function** if

1. $\operatorname{dom} J$ is a convex set.
2. For all $u, v \in \operatorname{dom} J$ and $\alpha \in [0, 1]$ it holds that

$$J(\alpha u + (1 - \alpha)v) \leq \alpha J(u) + (1 - \alpha)J(v).$$

We say J is **strictly convex** if the above inequality is strict for all $\alpha \in (0, 1)$ and $u \neq v$.

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Examples

- $J_{data}(u) = \|u - f\|_q^q$ where $q \geq 1$ and $\|\cdot\|_q$ is the ℓ^q-norm.
- $J_{regu}(u) = \|Ku\|_q^q$ where K is linear transform.
- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$ where $\alpha > 0$.

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- **Indicator function**:

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise}, \end{cases}$$

  where C is a convex subset of $\mathbb{E}$.

- Alternative formulation of constrained optimization:

$$\min J(u) \text{ over } u \in C. \quad \Leftrightarrow \quad \min J(u) + \delta_C(u) \text{ over } u \in \mathbb{E}.$$

Basic facts

(As exercises)

- Any norm (over a normed vector space) is a convex function.
- J is a convex function and K is a linear transform $\Rightarrow J(K\cdot)$ is convex function.
- (Jensen's inequality) $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex iff

$$J(\sum_{i=1}^{n} \alpha_i u^i) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

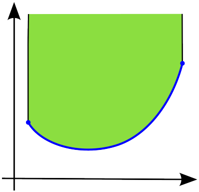
whenever $\{u^i\}_{i=1}^n \subset \mathbb{E}$, $\{\alpha_i\}_{i=1}^n \subset [0, 1]$, $\sum_{i=1}^n \alpha_i = 1$.

Epigraph

Definition

The **epigraph** of a proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{epi} J = \{(u, \alpha) \in \mathbb{E} \times \mathbb{R} : J(u) \le \alpha\}.$$



epigraph

Theorem

A proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex (resp. strictly convex) iff $\operatorname{epi} J$ is a convex (resp. strictly convex) set.

Proof: as exercise.

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Lipschitz continuity

Definition

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ with $\operatorname{ri}\operatorname{dom} J \neq \emptyset$. We say J is **locally Lipschitz** at $u \in \operatorname{ri}\operatorname{dom} J$ with modulus $L_u > 0$ if there exists $\epsilon > 0$ s.t.

$$|J(u^1) - J(u^2)| \leq L_u \|u^1 - u^2\| \quad \forall u^1, u^2 \in B_\epsilon(u) \cap \operatorname{ri}\operatorname{dom} J.$$

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Theorem

A proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \operatorname{ri}\operatorname{dom} J$.

Proof: on board.

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Global vs. Local minimizers

Recall the optimization of $J : \mathbb{E} \to \overline{\mathbb{R}}$:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Definition

1. $u^* \in \mathbb{E}$ is a **global minimizer** if $J(u^*) \leq J(u)$ for all $u \in \mathbb{E}$.
2. u^* is a **local minimizer** if $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u)$ for all $u \in B_\epsilon(u^*)$.
3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by strict inequality $J(u^*) < J(u)$.

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3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by strict inequality $J(u^*) < J(u)$.

Theorem

For any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, if $u^* \in \operatorname{dom} J$ is a local minimizer of J, then it is also a global minimizer.

Proof: on board.

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Existence of Minimizer

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Does a minimizer always exist?

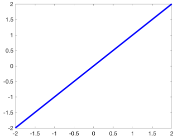
- Consider

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E},$$

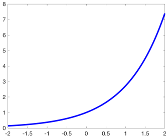
where $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a proper, convex function.

- Some counterexamples for $J : \mathbb{R} \to \overline{\mathbb{R}}$:

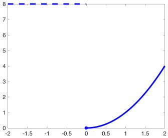
u



$\exp u$



$u^2 + \delta\{u > 0\}$



- We shall formalize our observations and derive sufficient conditions for existence.

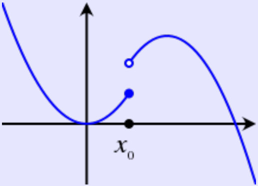
Sufficient conditions for existence

Definition

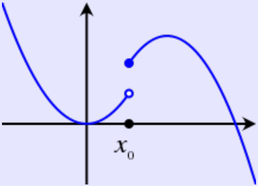
1. J is **bounded from below** if $J(\cdot) \geq C$ for some $C \in \mathbb{R}$.
2. J is **coercive** if $J(u) \to \infty$ whenever $\|u\| \to \infty$.
   - Proposition: J is coercive if $\operatorname{dom} J$ is bounded.
3. J is **lower-semicontinuous** (l.s.c.) if

$$J(u^*) \leq \liminf_{u \to u^*} J(u).$$

l.s.c.



not l.s.c.



- Proposition: J is l.s.c. iff $\operatorname{epi} J$ is closed.

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Theorem

Any proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$, which is bounded from below, coercive, and l.s.c., has a (global) minimizer.

Proof: on board.

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Remarks for infinite dimensions

1. Weak compactness in reflexive Banach (e.g. Hilbert) sp.
2. J is convex and strongly l.s.c. $\Rightarrow$ J is weakly l.s.c.

Uniqueness

- Recall that a function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is strictly convex if

$$J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v),$$

for all $u, v \in \operatorname{dom} J,\ u \neq v,\ \alpha \in (0, 1)$.

Theorem

The minimizer of a strictly convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is unique.

Proof: on board.

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Subdifferential

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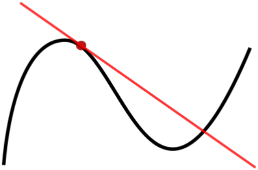
Differential



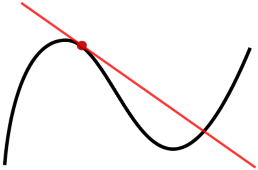
Definition

$J : \mathbb{E} \to \overline{\mathbb{R}}$ is called (Fréchet) **differentiable** at $u \in \operatorname{ri}\operatorname{dom} J$ and $\nabla J(u) \in \mathbb{E}$ is the (Fréchet) **differential** of J at u if

$$\lim_{h \to 0,\, h \in (\operatorname{ri}\operatorname{dom} J) - u} \frac{J(u+h) - J(u) - \langle \nabla J(u), h \rangle}{\|h\|} = 0.$$

J is said **continuously differentiable** at $u \in \operatorname{ri}\operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $\operatorname{dom} J \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Differential



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J is said **continuously differentiable** at $u \in \operatorname{ri}\operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $\operatorname{dom} J \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Remark

In functional analysis, $\nabla J(u)$ is treated as a *dual* object in $\mathbb{E}^*$, and $\langle \nabla J(u), h \rangle_{\mathbb{E}^*,\mathbb{E}}$ as *duality pairing*.