

Proofs in SS'17 Convex Optimization Lectures*

Last updated: May 17, 2017

1 Convex Analysis

Theorem 1.1 (separation of convex sets). *Let C_1, C_2 be nonempty convex subsets in \mathbb{E} such that $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then there exists a hyperplane separating C_1 and C_2 , i.e. $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ such that*

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1, u^2 \in C_2.$$

Proof. (i) Claim: Let $C \subset \mathbb{E}$ be closed, convex set, and $w \in \mathbb{E} \setminus C$. Then $\exists v \in \mathbb{E}, v \neq 0, \alpha \in \mathbb{R}$ s.t. $\langle v, w \rangle > \alpha \geq \langle v, u \rangle \forall u \in C$.

Consider the projection of w onto C , i.e. set $u^* := \arg \min_{u \in C} \frac{1}{2} \|u - w\|^2$ or, equivalently, let $\langle u - u^*, u^* - w \rangle \geq 0 \forall u \in C$.

Now set $v := w - u^* \neq 0$. Then $\forall u \in C$, we have $\langle v, w \rangle = \langle w - u^*, w \rangle = \|w - u^*\|^2 + \langle w - u^*, u^* \rangle \geq \|w - u^*\|^2 + \langle w - u^*, u \rangle = \|v\|^2 + \langle v, u \rangle$. Set $\alpha := \sup\{\langle v, u \rangle : u \in C\}$. Note $\alpha < \infty$ since $\langle v, u \rangle \leq \langle v, u^* \rangle \forall u \in C$. Thus $\langle v, w \rangle > \alpha \geq \langle v, u \rangle \forall u \in C$, which proves the claim.

(ii) Assume $C_2 = \{\bar{w}\}$ with $\bar{w} \in C_1$. Since $\mathbb{E} \setminus C_1$ is closed, $\exists w^k \in \mathbb{E} \setminus C_1$ s.t. $w^k \rightarrow \bar{w}$. For each w^k , by (i), $\exists v^k \in \mathbb{E}$ with $\|v^k\| = 1$ s.t. $\langle v^k, w^k \rangle \leq \langle v^k, u^1 \rangle \forall u^1 \in C_1$. Hence $v^k \rightarrow \bar{v} \in \mathbb{E}$ along a subsequence s.t. $\|\bar{v}\| = 1$ and $\langle \bar{v}, \bar{w} \rangle \leq \langle \bar{v}, u^1 \rangle \forall u^1 \in C_1$.

(iii) Consider C_2 as a general convex subset of \mathbb{E} . Set $C := C_2 - C_1 = \{u^2 - u^1 : u^1 \in C_1, u^2 \in C_2\}$. Note that C is a convex, open set, and $0 \in C$. By (ii), $\exists \bar{v}$ with $\|\bar{v}\| = 1$ s.t. $\langle -\bar{v}, u^2 - u^1 \rangle \geq \langle -\bar{v}, 0 \rangle = 0$ or, equivalently, $\langle \bar{v}, u^1 \rangle \geq \langle \bar{v}, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$. Set $\alpha := \sup\{\langle \bar{v}, u^2 \rangle : u^2 \in C_2\}$, then we conclude that $\langle \bar{v}, u^1 \rangle \geq \alpha \geq \langle \bar{v}, u^2 \rangle \forall u^1 \in C_1, u^2 \in C_2$. \square

Theorem 1.2. *A proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \text{ri dom } J$.*

Proof. (i) Claim: If $\sup\{J(v) : v \in B_\epsilon(u)\} < \infty$ for some $\epsilon > 0$, then J is locally Lipschitz at u .

Let $M := \sup\{J(v) : v \in B_\epsilon(u)\} < \infty$. By convexity of J , $\forall v \in B_\epsilon(u) : J(v) \geq 2J(u) - J(2u - v) \geq 2J(u) - M$. Thus, $\|J\|_{B_\epsilon(u)} := \sup\{|J(v)| : v \in B_\epsilon(u)\} \leq M + 2|J(u)|$.

Next we show J is Lipschitz on $B_{\epsilon/2}(u)$. Let $v, w \in B_{\epsilon/2}(u)$ be given. Take $z \in B_\epsilon(u)$ s.t. $w = (1 - t)v + tz$ for some $t \in [0, 1]$ and $\|z - v\| \geq \epsilon/2$. By convexity, $J(w) - J(v) \leq t(J(z) - J(v)) \leq 2t(M - J(u))$. Since $t(z - v) = w - v$, we have $t \leq \|w - v\| / \|z - v\| \leq 2\|w - v\| / \epsilon$ and $J(w) - J(v) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Analogously, one can show $J(v) - J(w) \leq (4(M - J(u))/\epsilon)\|w - v\|$. Hence, J is Lipschitz on $B_{\epsilon/2}(u)$ with modulus $4(M - J(u))/\epsilon$.

(ii) Let $u \in \text{ri dom } J$ and n be the dimension of the affine hull of $\text{dom } J$, then $\exists \{\alpha^i\}_{i=1}^{n+1} \subset (0, 1), \{u^i\}_{i=1}^{n+1} \subset \text{dom } J$ s.t. $u = \sum_{i=1}^{n+1} \alpha^i u^i, \sum_{i=1}^{n+1} \alpha^i = 1$, i.e. u belongs to the interior of the convex hull of $\{u^i\}_{i=1}^{n+1}$. Thus one can apply (i) to assert that J is locally Lipschitz at u . \square

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Theorem 1.3. For any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, if $u^* \in \text{dom } J$ is a local minimizer of J , then it is also a global minimizer.

Proof. By the definition of a local minimizer, $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u) \forall u \in B_\epsilon(u^*)$. For the sake of contradiction, assume $\exists \bar{u} \in \mathbb{E}$ s.t. $J(\bar{u}) < J(u^*)$. By convexity of J , we have $J(\alpha \bar{u} + (1 - \alpha)u^*) \leq J(u^*) - \alpha(J(u^*) - J(\bar{u})) \forall \alpha \in [0, 1]$. This violates the local optimality of u^* as $\alpha \rightarrow 0^+$. \square

Theorem 1.4. Any proper function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc, has a (global) minimizer.

Proof. Let $\{u^k\}$ be an infimizing sequence for J , i.e. $\lim_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u) > -\infty$. Since $\{J(u^k)\}$ is uniformly bounded from above, by coercivity of J , $\{u^k\}$ is uniformly bounded. By compactness, $u^k \rightarrow u^*$ along a subsequence. Since J is lsc, we have $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k) = \inf_{u \in \mathbb{E}} J(u)$, which implies $J(u^*) = \inf_{u \in \mathbb{E}} J(u)$ or u^* is a minimizer of J . \square

Theorem 1.5. The minimizer of a strictly convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is unique.

Proof. Let $u, v \in \mathbb{E}$ be two (global) minimizers s.t. $u \neq v$ and $J(u) = J(v) = J^*$. By strict convexity of J , $J(\alpha u + (1 - \alpha)v) < \alpha J(u) + (1 - \alpha)J(v) = J^*$ for all $\alpha \in (0, 1)$, which contradicts the global optimality of u and v . \square

Theorem 1.6. Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then ∂J is a monotone operator, i.e. $\forall u^1, u^2 \in \text{dom } J, \xi^1 \in \partial J(u^1), \xi^2 \in \partial J(u^2)$:

$$\langle \xi^1 - \xi^2, u^1 - u^2 \rangle \geq 0.$$

Proof. By applying the definition of subdifferential at arbitrarily given $u^1, u^2 \in \text{dom } J$, we have

$$\begin{aligned} J(u^2) &\geq J(u^1) + \langle \xi^1, u^2 - u^1 \rangle, \\ J(u^1) &\geq J(u^2) + \langle \xi^2, u^1 - u^2 \rangle. \end{aligned}$$

Adding the two inequalities yields $\langle \xi^1 - \xi^2, u^1 - u^2 \rangle \geq 0$. \square

Theorem 1.7. Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \text{int dom } J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

Proof. (i) nonemptiness. Since $(u, J(u)) \notin \text{int epi } J$, by Theorem 1.1, $\exists (\xi, -\alpha) \in \mathbb{E} \times \mathbb{R}$ s.t. $(\xi, -\alpha) \neq (0, 0)$, $\alpha \geq 0$ by our choice, and $\langle (\xi, -\alpha), (u - v, J(u) - J(v)) \rangle \geq 0 \forall v \in \text{dom } J$. In fact, we must have $\alpha > 0$ since otherwise $\xi = 0$. Thus, we conclude that $\xi/\alpha \in \partial J(u)$.

(ii) boundedness. By Theorem 1.2, J is locally Lipschitz at u with modulus L_u . Let $\xi \in \partial J(u)$ be fixed. For any $d \in (\text{dom } J) - u$ with $\|d\|$ sufficiently small, we have $\langle \xi, d \rangle \leq J(u + d) - J(u) \leq L_u \|d\|$. This holds true only if $\|\xi\| \leq L_u$, which implies boundedness of $\partial J(u)$.

(iii) closedness. Let $v \in \mathbb{E}$ be fixed and $\xi^k \rightarrow \xi^*$ where each $\xi^k \in \partial J(u)$. Then $\forall k : J(v) - J(u) \geq \langle \xi^k, v - u \rangle$. By continuity, $J(v) - J(u) \geq \langle \xi^*, v - u \rangle$ in the limit. Since v can be arbitrary, we assert $\xi^* \in \partial J(u)$.

(iv) convexity. Let $v \in \mathbb{E}$ be arbitrarily given, $\xi, \eta \in \partial J(u)$. Then we have

$$\begin{aligned} J(v) &\geq J(u) + \langle \xi, v - u \rangle, \\ J(v) &\geq J(u) + \langle \eta, v - u \rangle. \end{aligned}$$

Hence, $\forall 0 \leq \alpha \leq 1 : J(v) \geq J(u) + \langle \alpha \xi + (1 - \alpha)\eta, v - u \rangle$, i.e. $\alpha \xi + (1 - \alpha)\eta \in \partial J(u)$. \square

Theorem 1.8. *Let $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then ∂J is a closed set-valued map, i.e. $\xi^* \in \partial J(u^*)$ whenever*

$$\exists(u^k, \xi^k) \rightarrow (u^*, \xi^*) \in (\text{ri dom } J) \times \mathbb{E} \text{ s.t. } \xi^k \in \partial J(u^k) \forall k.$$

Proof. Let $v \in \mathbb{E}$ be fixed. For each k , $\xi^k \in \partial J(u^k) \Rightarrow J(v) \geq J(u^k) + \langle \xi^k, v - u^k \rangle$. Passing $k \rightarrow \infty$, we have $\langle \xi^k, v - u^k \rangle \rightarrow \langle \xi^*, v - u^* \rangle$ and $J(u^*) \leq \liminf_{k \rightarrow \infty} J(u^k)$. Hence, $J(u^*) + \langle \xi^*, v - u^* \rangle \leq \liminf_{k \rightarrow \infty} \{J(u^k) + \langle \xi^k, v - u^k \rangle\} \leq J(v)$. Since v can be arbitrary, $\xi^* \in \partial J(u^*)$. \square

Theorem 1.9. *Given any proper convex function $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is: $0 \in \partial J(u^*)$.*

Proof. (i) sufficiency. $0 \in \partial J(u^*) \Rightarrow J(u) \geq J(u^*) + \langle 0, u - u^* \rangle = J(u^*) \forall u \in \mathbb{E}$.

(ii) necessity. $J(u^*) \leq J(u) \forall u \in \mathbb{E} \Rightarrow J(u^*) + \langle 0, u - u^* \rangle \leq J(u) \forall u \Rightarrow 0 \in \partial J(u^*)$. \square

Theorem 1.10 (Fenchel-Young inequality). *For all $u \in \text{dom } J$, $p \in \text{dom } J^*$, we have $J(u) + J^*(p) \geq \langle u, p \rangle$. The equality holds iff $p \in \partial J(u)$.*

Proof. $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate; $p \in \partial J(u)$ is the sufficient and necessary condition for: $u = \arg \min_{v \in \mathbb{E}} J(v) - \langle v, p \rangle$. \square

Theorem 1.11. *Assume $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ and $J^{**} = (J^*)^*$ is the biconjugate of J . In general:*

1. $J^{**}(\cdot) \leq J(\cdot)$.

2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.

4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof. (1) Since $J^{**}(u) = \sup_p \langle p, u \rangle - J^*(p)$ and $\langle p, u \rangle - J^*(p) \leq J(u)$ by Theorem 1.10, we assert $J^{**}(\cdot) \leq J(\cdot)$.

(2) (i) convexity. Let $p, q \in \mathbb{E}$, $0 \leq \alpha \leq 1$. Then $J^*(\alpha p + (1-\alpha)q) = \sup_u \{\langle u, \alpha p + (1-\alpha)q \rangle - J(u)\} \leq \sup_u \{\langle \alpha u, p \rangle - \alpha J(u)\} + \sup_u \{\langle (1-\alpha)u, q \rangle - (1-\alpha)J(u)\} = \alpha J^*(p) + (1-\alpha)J^*(q)$.

(ii) lsc. Note $\text{epi } J^* = \{(p, \alpha) \in \mathbb{E} \times \mathbb{R} : \langle u, p \rangle - J(u) \leq \alpha \forall u\} = \bigcap_u \text{epi } \Phi_u$ where $\Phi_u(\cdot) = \langle u, \cdot \rangle - J(u)$. Since each $\text{epi } \Phi_u$ and any arbitrary intersection of closed sets is closed, $\text{epi } J^*$ is closed and hence J^* is lsc.

(3) For the sake of contradiction, assume $\exists \bar{u} \in \text{dom } J^{**}$ s.t. $J(\bar{u}) > J^{**}(\bar{u})$. Let $d := \frac{1}{2}(J(\bar{u}) - J^{**}(\bar{u})) > 0$. Since $(\bar{u}, J(\bar{u}) - d) \notin \text{epi } J$ and $\text{epi } J$ is closed, by Theorem 1.1, $\exists(\bar{p}, -1) \in \mathbb{E} \times \mathbb{R}$ s.t. $\langle (\bar{p}, -1), (\bar{u}, J(\bar{u}) - d) \rangle \geq \langle (\xi, -1), (u, \alpha) \rangle \forall (u, \alpha) \in \text{epi } J$. In particular, $\langle \bar{p}, \bar{u} \rangle - J(\bar{u}) + d \geq \langle \bar{p}, \bar{u} \rangle - J(\bar{u}) \forall u$. Hence, $\langle \bar{p}, \bar{u} \rangle - J(\bar{u}) + d \geq J^*(\bar{p}) \geq \langle \bar{p}, \bar{u} \rangle - J^{**}(\bar{u})$ by Theorem 1.10. Thus we have $J^{**}(\bar{u}) \geq J(\bar{u})$ as a contradiction to our assumption.

- (4) $p \in \partial J(u) \Leftrightarrow J(u) + J^*(p) = \langle u, p \rangle \Leftrightarrow J^{**}(u) + J^*(p) = \langle u, p \rangle \Leftrightarrow u \in \partial J^*(p)$. \square

Theorem 1.12. *Assume that $J : \mathbb{E} \rightarrow \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ -strongly convex iff J^* has $\frac{1}{\mu}$ -Lipschitz gradient.*

Proof. Let $p \in \partial J(u)$ be arbitrarily given. By μ -strong convexity of J , we have

$$J(v) \geq J(u) + \langle p, v - u \rangle + \frac{\mu}{2} \|v - u\|^2 \quad \forall v. \quad (1)$$

Then $\forall q : J^*(q) = \sup_v \{\langle q, v \rangle - J(v)\} \leq \sup_v \{\langle q, v \rangle - J(u) - \langle p, v - u \rangle - \frac{\mu}{2} \|v - u\|^2\} = \langle q, u \rangle - J(u) + \sup_v \{\langle q - p, v - u \rangle - \frac{\mu}{2} \|v - u\|^2\} = \langle q, u \rangle - J(u) + \frac{1}{2\mu} \|q - p\|^2 = \langle p, u \rangle - J(u) + \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2 = J^*(p) \langle q - p, u \rangle + \frac{1}{2\mu} \|q - p\|^2$. Here we have used the identity $\langle p, u \rangle - J(u) = J^*(p)$. We have actually derived $\lim_{q \rightarrow p} \|J^*(q) - J^*(p) - \langle q - p, u \rangle\| / \|q - p\| = 0$, which asserts that J^* is (Fréchet-)differentiable at p with $\nabla J^*(p) = u$.

Finally we show ∇J^* is $\frac{1}{\mu}$ -Lipschitz. Let $u = \nabla J^*(p)$, $v = \nabla J^*(q)$, or equivalently $p \in \partial J(u)$, $q \in \partial J(v)$. Then by (1) we have

$$\begin{aligned} J(v) &\geq J(u) + \langle p, v - u \rangle + \frac{\mu}{2} \|v - u\|^2, \\ J(u) &\geq J(v) + \langle q, u - v \rangle + \frac{\mu}{2} \|u - v\|^2. \end{aligned}$$

Adding the above two inequalities, we obtain $\mu \|u - v\|^2 \leq \langle p - q, u - v \rangle \leq \|p - q\| \|u - v\|$ and thus $\|u - v\| \leq \frac{1}{\mu} \|p - q\|$. \square

Theorem 1.13 (Fenchel-Rockafellar duality). *Assume $\exists \bar{u} \in \text{dom } G$ s.t. F is continuous at $K\bar{u}$. Then the strong duality holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal solution pair iff*

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof. Define $\Phi(\cdot) := \inf_u \{F(Ku + \cdot) + G(u)\}$. Since $\forall v^1, v^2 \in \mathbb{R}^m$, $\forall \alpha \in [0, 1] : \alpha \Phi(v^1) + (1 - \alpha) \Phi(v^2) = \inf_{u^1} \{\alpha F(Ku^1 + v^1) + \alpha G(u^1)\} + \inf_{u^2} \{(1 - \alpha) F(Ku^2 + v^2) + (1 - \alpha) G(u^2)\} = \inf_{u^1, u^2} \{\alpha F(Ku^1 + v^1) + (1 - \alpha) F(Ku^2 + v^2) + \alpha G(u^1) + (1 - \alpha) G(u^2)\} \geq \inf_u \{\alpha F(Ku + v^1) + (1 - \alpha) F(Ku + v^2) + G(u)\} \geq \inf_u \{F(Ku + \alpha v^1 + (1 - \alpha)v^2) + G(u)\} = \Phi(\alpha v^1 + (1 - \alpha)v^2)$, we prove that Φ is a convex function.

Without loss of generality, assume $\Phi(0) > -\infty$. By our assumption, $\exists \epsilon > 0$ s.t. $\forall \|v\| < \epsilon : \Phi(v) \leq F(K\bar{u} + v) + G(\bar{u}) \leq M$ for some $M < \infty$. Hence, by Theorem 1.2, Φ is locally Lipschitz at 0, and $\Phi(0) = \Phi^{**}(0) = \sup_p -\Phi^*(p)$, where $\Phi^*(p) = \sup_v \{\langle p, v \rangle - \inf_u \{F(Ku + v) + G(u)\}\} = \sup_{v, u} \{\langle p, v + Ku \rangle + \langle -K^\top p, u \rangle - F(Ku + v) - G(u)\} = F^*(p) + G^*(-K^\top p)$. Thus, $\mathcal{P}^* = \mathcal{D}^*$ is proven.

As for the optimality condition, note that $\forall (u, p) : \mathcal{G}(u, p) = F(Ku) + G(u) + G^*(-K^\top p) + F^*(p) = F(Ku) + F^*(p) - \langle Ku, p \rangle + G(u) + G^*(-K^\top p) - \langle -K^\top p, u \rangle \geq 0$. The equality holds, i.e. $\mathcal{G}(u^*, p^*) = 0$, iff $Ku^* \in \partial F^*(p^*)$ and $-K^\top p^* \in \partial G(u^*)$ according to Theorem 1.10. \square