Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 9

Room: 02.09.023

Monday, 10.07.2017, 12:15-14:00

Submission deadline: Wednesday, 05.07.2017, Room 02.09.023

Monotone Operators (6 Points)

Exercise 1 (6 Points)**.** Prove the theorem from the lecture:

Let C be a nonempty, closed, convex subset of $\mathbb{R}^n$. For each $i \in \{1, ..., m\}$, let $\alpha_i \in (0,1)$, $\omega_i \in (0,1)$ and $\Phi_i : C \to \mathbb{R}^n$ be an α_i-averaged operator. If $\sum_{i=1}^m \omega_i = 1$ and $\alpha = \max_{1 \le i \le m} \alpha_i$, then

$$\Phi = \sum_{i=1}^m \omega_i \Phi_i$$

is α-averaged.

Solution. Φ_i is α_i-averaged iff

$$\|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha_i}{\alpha_i}\|(I - \Phi_i)(u) - (I - \Phi_i)(v)\|_2^2 \le \|u - v\|_2^2,$$

for all $u, v \in C$. We have the estimate

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_2^2 + \frac{1-\alpha}{\alpha}\|(I - \Phi)(u) - (I - \Phi)(v)\|_2^2 \\
&= \left\|\sum_{i=1}^m \omega_i(\Phi_i(u) - \Phi_i(v))\right\|_2^2 + \frac{1-\alpha}{\alpha}\left\|\left(I - \sum_{i=1}^m \omega_i\Phi_i\right)(u) - \left(I - \sum_{i=1}^m \omega_i\Phi_i\right)(v)\right\|_2^2 \\
&\le \sum_{i=1}^m \omega_i \|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha}{\alpha}\left\|\sum_{i=1}^m \omega_i((I - \Phi_i)(u) - (I - \Phi_i)(v))\right\|_2^2 \\
&\le \sum_{i=1}^m \omega_i \|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha}{\alpha}\sum_{i=1}^m \omega_i \|(I - \Phi_i)(u) - (I - \Phi_i)(v)\|_2^2 \\
&= \sum_{i=1}^m \omega_i \left(\|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha}{\alpha}\|(I - \Phi_i)(u) - (I - \Phi_i)(v)\|_2^2\right).
\end{aligned}$$

Since $1 > \alpha \geq \alpha_i > 0$ for all i we have that $\frac{1}{\alpha} - 1 \leq \frac{1}{\alpha_i} - 1$. Then we can further bound:

$$\begin{aligned}
\ldots &= \sum_{i=1}^{m} \omega_i \left(\|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha}{\alpha} \|(I - \Phi_i)(u) - (I - \Phi_i)(v)\|_2^2 \right) \\
&\leq \sum_{i=1}^{m} \omega_i \left(\|\Phi_i(u) - \Phi_i(v)\|_2^2 + \frac{1-\alpha_i}{\alpha_i} \|(I - \Phi_i)(u) - (I - \Phi_i)(v)\|_2^2 \right) \\
&\leq \sum_{i=1}^{m} \omega_i \|u - v\|_2^2 = \|u - v\|_2^2.
\end{aligned}$$

Programming

Exercise 2. Finish the programming exercise from last week. The exact deadline for handing in the programming exercise is **July 5th, 2017, 23:59pm**.