



Multiple View Geometry: Exercise Sheet 1

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Part I: Theory

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span \mathbb{R}^3 and (3) whether they form a basis of \mathbb{R}^3 :

$$(a) B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The set B_1 (1) is linearly independent, (2) spans \mathbb{R}^3 , (3) forms a basis of \mathbb{R}^3 .

This can be shown by building a matrix and calculating the determinant:

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0.$$

As the determinant is not zero, we know that the vectors are linearly independent. Three linearly independent vectors in \mathbb{R}^3 span \mathbb{R}^3 . A set is a basis of \mathbb{R}^3 if it is linearly independent and spans \mathbb{R}^3 , so B_1 forms a basis.

$$(b) B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The set B_2 (1) is linearly independent, (2) does not span \mathbb{R}^3 , (3) does not form a basis of \mathbb{R}^3 .

Since the two vectors are not parallel, linear independence is given. To span \mathbb{R}^3 , there are at least three vectors needed. Hence, the set cannot be a basis either.

$$(c) B_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The set B_3 (1) is not linearly independent, (2) spans \mathbb{R}^3 , (3) does not form a basis of \mathbb{R}^3 . In \mathbb{R}^3 , there cannot be more than three independent vectors. Using e.g. the determinant, one finds that any three of the four vectors form a basis of \mathbb{R}^3 and thus the four together span \mathbb{R}^3 . Since they are not linearly independent, they cannot form a basis.

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

$$(a) G_1 := \{A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \wedge A^\top = A\}$$

The set is not closed under multiplication, thus no group. To show this, one counterexample is enough: choose $n = 3$ and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \in G_1, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in G_1 : \quad AB = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 12 \\ 3 & 8 & 15 \end{pmatrix} \notin G_1$$

Note (comment to a question asked during the exercise):

You can also show that if G_1 was a group, for any $A, B \in G_1$, $(AB)^\top = AB$ would have to be true, but is not. This is equivalent to saying $BA = AB$ would have to be true:

$$(AB)^\top = B^\top A^\top = BA$$

However, to show that there exist A and B in G_1 for which $AB \neq BA$ (which is an important step in the proof!), the easiest way again is to choose a concrete counter-example.

(b) $G_2 := \{A \in \mathbb{R}^{n \times n} \mid \det(A) = -1\}$

The set contains no neutral element, thus no group:

$$\det(\text{Id}_n) = 1 \neq -1 \quad \Rightarrow \quad \text{Id}_n \notin G_2$$

(c) $G_3 := \{A \in \mathbb{R}^{n \times n} \mid \det(A) > 0\}$

The set forms a group. The easiest way to show this is to show that G_3 is a subgroup of the general linear group $GL(n)$. We simply need to show that for any two elements A, B of G_3 , AB^{-1} is also in G_3 :¹ for $A, B \in G_3$,

$$\det(AB^{-1}) = \underbrace{\det(A)}_{>0} \underbrace{[\det(B)]^{-1}}_{>0} > 0 \quad \Rightarrow \quad AB^{-1} \in G_3$$

Thus, G_3 is a subgroup of $GL(n)$ and hence a group.

3. Prove or disprove: There exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, \dots, 5 : \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

Assume there exist five pairwise orthogonal, non-zero vectors $\mathbf{v}_1, \dots, \mathbf{v}_5 \in \mathbb{R}^3$. In \mathbb{R}^3 , there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists a_i : \quad \sum_{i=1}^5 a_i \mathbf{v}_i = \mathbf{0},$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$\mathbf{v}_1 = a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$\begin{aligned} \|\mathbf{v}_1\|^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \\ &= \langle \mathbf{v}_1, a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + a_4 \mathbf{v}_4 + a_5 \mathbf{v}_5 \rangle = \\ &= a_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + a_3 \langle \mathbf{v}_1, \mathbf{v}_3 \rangle + a_4 \langle \mathbf{v}_1, \mathbf{v}_4 \rangle + a_5 \langle \mathbf{v}_1, \mathbf{v}_5 \rangle = \\ &= 0 + 0 + 0 + 0 = 0 \\ \Rightarrow \quad \mathbf{v}_1 &= \mathbf{0}, \end{aligned}$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

¹See e.g. https://en.wikipedia.org/wiki/Subgroup_test for a proof if this is not clear to you.