Multiple View Geometry: Exercise Sheet 8

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Part I: Theory

1. **Image Warping**

(a) Look at the warping function $\tau(\xi, \mathbf{x})$ in Eq. (9). What do $\tau(\xi, \mathbf{x})$ and $r_i(\xi)$ look like at $\xi = \mathbf{0}$?

For $\xi = \mathbf{0}$, we have

$$T(g(\mathbf{0}), \mathbf{p}) = T((\mathrm{Id}_3, \mathbf{0}), \mathbf{p}) = \mathrm{Id}_3\mathbf{p} + \mathbf{0} = \mathbf{p}\,.$$

Thus, $\tau(\mathbf{0}, \mathbf{x})$ becomes

$$\tau(\mathbf{0}, \mathbf{x}) = \pi\left(\pi^{-1}(\mathbf{x}, Z_1(\mathbf{x}))\right) = \mathbf{x}\,,$$

where the last equality follows from inserting the formulas for π and π^{-1}. Finally,

$$r_i(\mathbf{0}) = I_2(\tau(\mathbf{0}, \mathbf{x}_i)) - I_1(\mathbf{x}_i) = I_2(\mathbf{x}_i) - I_1(\mathbf{x}_i)\,.$$

(b) Prove that the derivative of $r_i(\xi)$ w.r.t. ξ at $\xi = \mathbf{0}$ is

$$\left.\frac{\partial r_i(\xi)}{\partial \xi}\right|_{\xi=\mathbf{0}} = \frac{1}{z}\begin{pmatrix} I_x f_x & I_y f_y \end{pmatrix} \left.\begin{pmatrix} 1 & 0 & -\frac{x}{z} & -\frac{xy}{z} & z + \frac{x^2}{z} & -y \\ 0 & 1 & -\frac{y}{z} & -z - \frac{y^2}{z} & \frac{xy}{z} & x \end{pmatrix}\right|_{(x,y,z)^\top = \pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i))}$$

To this end, apply the chain rule multiple times and use the following identity:

$$\left.\frac{\partial T\left(g(\xi), \mathbf{p}\right)}{\partial \xi}\right|_{\xi=\mathbf{0}} = \begin{pmatrix} \mathrm{Id}_3 & -\hat{\mathbf{p}} \end{pmatrix} \in \mathbb{R}^{3\times 6}\,.$$

Since $I_1(\mathbf{x}_i)$ does not depend on ξ, we only need to look at the first term in $r_i(\xi)$. It is a composition of the functions I_2, π and T. Applying the chain rule gives

$$\left.\frac{\partial r_i(\xi)}{\partial \xi}\right|_{\xi=\mathbf{0}} = \left.\frac{\partial I_2(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\pi(T(g(\mathbf{0}), \pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i))))} \cdot \left.\frac{\partial \pi(\mathbf{p})}{\partial \mathbf{p}}\right|_{\mathbf{p}=T(g(\mathbf{0}), \pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i)))} \cdot \left.\frac{\partial T(g(\xi), \pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i)))}{\partial \xi}\right|_{\xi=\mathbf{0}}\,.$$

Now, we know that $T(g(\mathbf{0}), \mathbf{p}) = \mathbf{p}$, so we can write

$$\left.\frac{\partial r_i(\xi)}{\partial \xi}\right|_{\xi=\mathbf{0}} = \left.\left[\left(\nabla I_2\left(\pi(\mathbf{p})\right)\right)^\top \cdot \frac{\partial \pi(\mathbf{p})}{\partial \mathbf{p}} \cdot \begin{pmatrix} \mathrm{Id}_3 & -\hat{\mathbf{p}} \end{pmatrix}\right]\right|_{\mathbf{p}=\pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i))}$$

The second and third term are

$$\frac{\partial \pi(\mathbf{p})}{\partial \mathbf{p}} = \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{f_x x}{z}\right) & \frac{\partial}{\partial y}\left(\frac{f_x x}{z}\right) & \frac{\partial}{\partial z}\left(\frac{f_x x}{z}\right) \\ \frac{\partial}{\partial x}\left(\frac{f_y y}{z}\right) & \frac{\partial}{\partial y}\left(\frac{f_y y}{z}\right) & \frac{\partial}{\partial z}\left(\frac{f_y y}{z}\right) \end{pmatrix} = \frac{1}{z}\begin{pmatrix} f_x & 0 \\ 0 & f_y \end{pmatrix}\begin{pmatrix} 1 & 0 & -\frac{x}{z} \\ 0 & 1 & -\frac{y}{z} \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{Id}_3 & -\hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \end{pmatrix}$$

Performing the matrix multiplication and using

$$\pi(\mathbf{p}) = \pi\left(\pi^{-1}(\mathbf{x}_i, Z_1(\mathbf{x}_i))\right) = \mathbf{x}_i$$

(see (a)) as well as

$$(\nabla I_2(\mathbf{x}_i))^\top \begin{pmatrix} f_x & 0 \\ 0 & f_y \end{pmatrix} = \begin{pmatrix} I_x(\mathbf{x}_i) f_x & I_y(\mathbf{x}_i) f_y \end{pmatrix}$$

leads to the desired result.

2. **Image Pyramids**

How does the camera matrix K change from level l to $l+1$? Write down $f_x^{(l+1)}$, $f_y^{(l+1)}$, $c_x^{(l+1)}$ and $c_y^{(l+1)}$ in terms of $f_x^{(l)}$, $f_y^{(l)}$, $c_x^{(l)}$ and $c_y^{(l)}$.

Looking at how each pixel coordinate transforms from one image level l to the next, $l+1$, we have

$$2\mathbf{x}^{(l+1)} + \tfrac{1}{2} = \mathbf{x}^{(l)} \quad \Rightarrow \quad \mathbf{x}^{(l+1)} = \tfrac{1}{2}\mathbf{x}^{(l)} - \tfrac{1}{4}\,.$$

Plugging into the relations $\bar{\mathbf{x}}^{(l)} = \frac{1}{Z}K^{(l)}\mathbf{X}$ and $\bar{\mathbf{x}}^{(l+1)} = \frac{1}{Z}K^{(l+1)}\mathbf{X}$ results in

$$f_x^{(l+1)} = \tfrac{1}{2}f_x^{(l)}\,, \quad f_y^{(l+1)} = \tfrac{1}{2}f_y^{(l)}\,, \quad c_x^{(l+1)} = \tfrac{1}{2}c_x^{(l)} - \tfrac{1}{4}\,, \quad c_y^{(l+1)} = \tfrac{1}{2}c_y^{(l)} - \tfrac{1}{4}\,.$$

3. **Optimization for Normally Distributed $p(r_i)$**

(a) Confirm that a normally distributed $p(r_i)$ with a uniform prior on the camera motion leads to normal least squares minimization. To this end, insert

$$p(r_i|\xi) = p(r_i) = A\exp\left(-\frac{r_i^2}{\sigma^2}\right)$$

into Eq. (15) (use $p(\xi) = \text{const}$ there) and show that

$$\xi_{\text{MAP}} = \arg\min_\xi \sum_i r_i(\xi)^2\,.$$

$$p(r_i|\xi) = p(r_i) = A\exp\left(-\frac{r_i^2}{\sigma^2}\right) \quad \Rightarrow \quad -\log p(r_i|\xi) = -\log A + \frac{r_i^2}{\sigma^2}$$

Inserting into Eq. (15) gives

$$\xi_{\text{MAP}} = \arg\min_\xi \left(-N\log A + \frac{1}{\sigma^2}\sum_i r_i(\xi)^2 - \log p(\xi)\right) = \arg\min_\xi \sum_i r_i(\xi)^2\,,$$

since $-N\log A$ and $-\log p(\xi)$ are just constant shifts and $\frac{1}{\sigma^2}$ is only a scaling, and none of them changes the argmin.

(b) Explicitly show that the weights

$$w(r_i) = \frac{1}{r_i}\frac{\partial \log p(r_i)}{\partial r_i}$$

are constant for normally distributed $p(r_i)$.

$$w(r_i) = \frac{1}{r_i}\frac{\partial \log p(r_i)}{\partial r_i} = \frac{1}{r_i}\frac{\partial\left(\log A - \frac{r_i(\xi)^2}{\sigma^2}\right)}{\partial r_i} = \frac{1}{r_i}\left(0 - \frac{2r_i}{\sigma^2}\right) = -\frac{2}{\sigma^2} = \text{const}(r_i)$$

(c) Show that in the case of normally distributed $p(r_i)$ the update step $\Delta\xi$ can be computed as

$$\Delta\xi = -\left(J^\top J\right)^{-1} J^\top \mathbf{r}(\mathbf{0})\ .$$

Eq. (21) reads

$$J^\top W J \Delta\xi = -J^\top W \mathbf{r}(\mathbf{0})\ ,$$

with W a diagonal matrix with constant diagonal entries $W_{ii} = w(r_i) = -\frac{2}{\sigma^2}$.

$$\Rightarrow \quad W = -\frac{2}{\sigma^2}\text{Id} \quad \Rightarrow \quad -\frac{2}{\sigma^2}J^\top J\Delta\xi = \frac{2}{\sigma^2}J^\top \mathbf{r}(\mathbf{0}) \quad \Rightarrow \quad \text{claim}\ .$$