Probabilistic Graphical Models in Computer Vision (IN2329)

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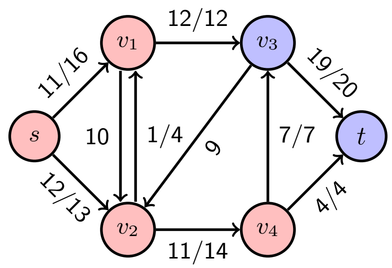
Agenda for today's lecture *

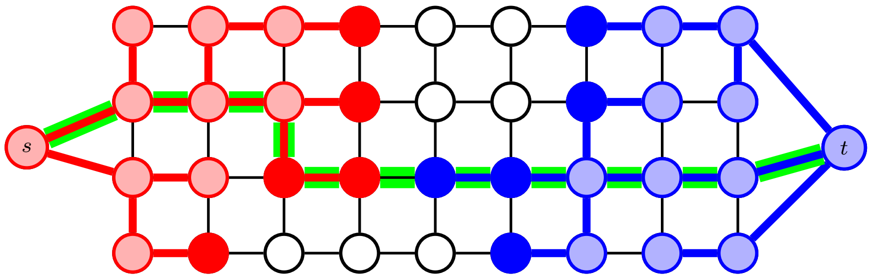
In the **previous lecture** we learnt about the minimum $s - t$ cut problem



Today we are going to learn about

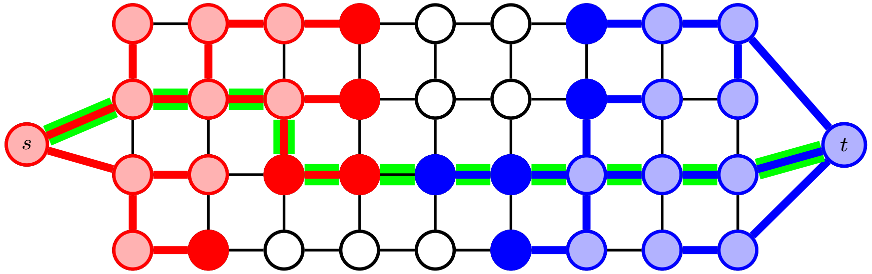
- The *Boykov–Kolmogorov algorithm*
- Exact solution for **binary image segmentation** via *graph cut*
- Approximate solution for the **multi-label problem** via $\alpha - \beta$ swap

Boykov–Kolmogorov algorithm



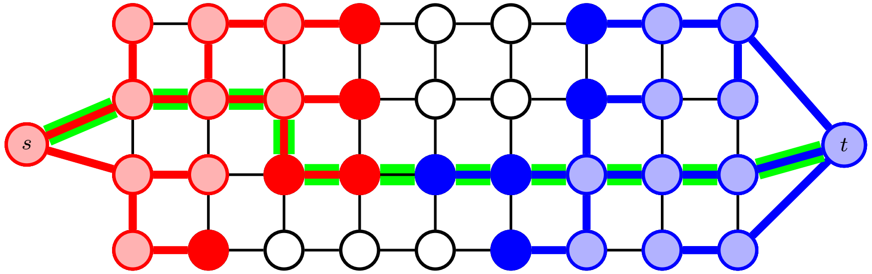
- Main idea: Never start building an *augmenting path* from scratch
- Two non-overlapping search trees S and T with roots at the terminals
- The edges of the trees are *non-saturated*, i.e. $f(i,j) < c(i,j)$
- Active nodes:
- Passive nodes:
- Free nodes:

Boykov–Kolmogorov algorithm



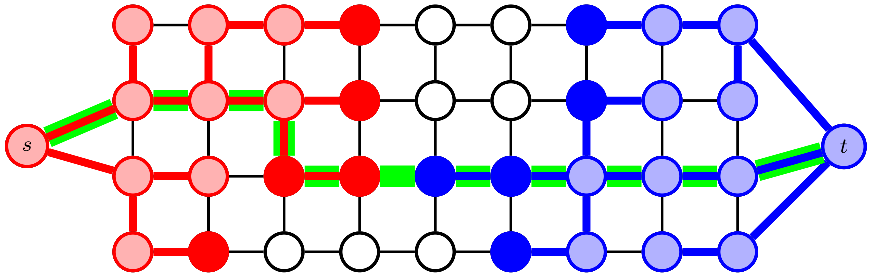
while `true` **do**
  grow S or T to find an augmenting path P from s to t
  if $P = \emptyset$ **then**
    `terminate`
  end if
  augment on P
  adopt orphans
end while

Growth stage



- The *active nodes* explore adjacent edges and acquire new children from a set of *free nodes*
- The newly acquired nodes become *active* members of the corresponding search trees
- The *active node* becomes *passive*, when all of its neighbors are explored
- If an *active node* encounters a neighboring node belonging to the opposite tree, the growth stage terminates

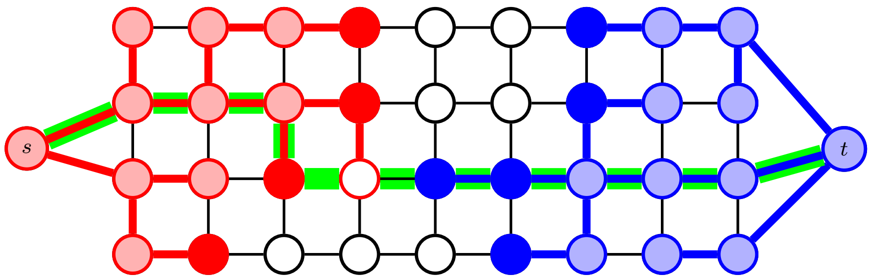
Augmentation stage



- Find the bottleneck capacity Δ on P
- Update the residual graph by pushing flow Δ through P

Adoption stage

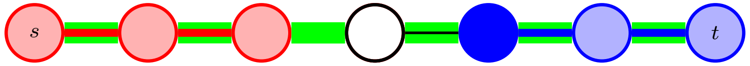
- *Orphan* ( ): the nodes such that the edges linking them to their parents are no longer valid (i.e. they are saturated)
- By removing them the search trees S and T may be split into *forests*



We are trying to find a *new valid parent* for p among its neighbors, such that a new parent should belong to the same set, S or T, as the *orphan*

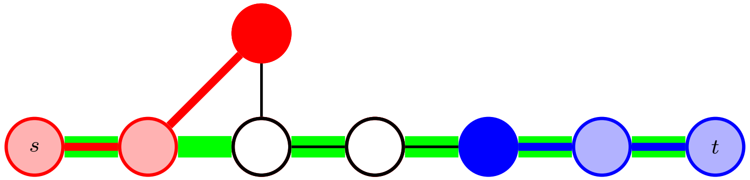
Adoption stage

If an orphan p does not find a valid parent then it becomes a *free node*



Scan all neighbors q of p such that q belong to the same tree as p:

- if tree $c(q, p) > 0$, add q to the *active set*
- if $\texttt{parent}(q) = p$, add q to the set of *orphans* and set $\texttt{parent}(q) = \varnothing$



Complexity

- The *Boykov-Kolmogorov algorithm* is also an *augmented path-based method* with worst case complexity $\mathcal{O}(|\mathcal{E}| \cdot |\mathcal{V}|^2 \cdot |C|)$, where $|C|$ is the capacity of the minimum cut.
- This complexity is worse than complexities of *Edmonds–Karp algorithm*, however, this algorithm *significantly* ($\sim$2-10$\times$) outperforms standard algorithms on typical problem instances in vision.

Regular functions *

Let us consider a function f of two binary variables, then f is called **regular**, if it satisfies the following inequality

$$f(0,0) + f(1,1) \leqslant f(0,1) + f(1,0)\ .$$

Example: the **Potts-model** is *regular*, since

$$[\![0 \neq 0]\!] + [\![1 \neq 1]\!] = 0 \leqslant 2 = [\![0 \neq 1]\!] + [\![1 \neq 0]\!]\ .$$

Regular energy functions

Let us consider an *energy function* E of n binary variables which can be written as the sum of functions of up to two variables, that is $E : \mathbb{B}^n \to \mathbb{R}$

$$E(y_1, \ldots, y_n) = \sum_i E_i(y_i) + \sum_{i<j} E_{ij}(y_i, y_j)\ .$$

E is *regular*, if each term $E_{ij} : \mathbb{B}^2 \to \mathbb{R}$ for all $i < j$ satisfies

$$E_{ij}(0,0) + E_{ij}(1,1) \leqslant E_{ij}(0,1) + E_{ij}(1,0)\ .$$

If each term E_{ij} is *regular*, then it is possible to find the **global** minimum of E in *polynomial time* by solving a *minimum $s-t$ cut problem*.

Binary image segmentation

We have already seen that **binary image segmentation** can be reformulated as the minimization of an *energy function* $E : \mathbb{B}^{\mathcal{V}} \times \mathcal{X} \to \mathbb{R}$:

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; x_i) + \sum_{(i,j) \in \mathcal{E}} w \cdot [\![y_i \neq y_j]\!] \ .$$

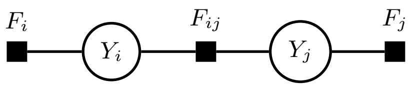
where $\mathcal{V}$ corresponds to the output variables, i.e. the pixels, and $\mathcal{E}$ includes the pairs of 4-neighboring pixels.

Assume probability densities f_{bg} and f_{fg} estimated for the background and the foreground, respectively. The **unary energies** E_i for all $i \in \mathcal{V}$ can be defined as

$$\begin{aligned} E_i(0, x_i) &= 0 \ , \\ E_i(1, x_i) &= \log \frac{f_{\mathsf{bg}}(x_i)}{f_{\mathsf{fg}}(x_i)} \ . \end{aligned}$$

Energy minimization via minimum $s - t$ cut

Let us consider the following example



Through this example we illustrate how to minimize *regular energy functions* consisting of up to pairwise relationships. In our example $\mathbf{y} \in \mathbb{B}^2$ and $E(\mathbf{y})$ is defined as

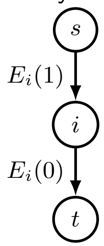
$$E(\mathbf{y}) = E_1(y_1) + E_2(y_2) + E_{12}(y_1, y_2) \ .$$

We will create a *flow network* $(\mathcal{V} \cup \{s, t\}, \mathcal{E}', c, s, t)$ such that **the minimum $s - t$ cut will correspond to the minimization of our energy function** $E(\mathbf{y})$, where the *labeling* for each $i \in \mathcal{V}$ is defined as

$$y_i = \begin{cases} 0, & \text{if } i \in \mathcal{S} \ , \\ 1, & \text{if } i \in \mathcal{T} \ . \end{cases}$$

Graph construction: unary energies

Let us consider the unary energy function $E_i : \{0,1\} \to \mathbb{R}$.

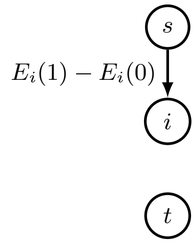


Obviously, the minimum $s-t$ cut of the flow network will correspond to

$$\operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) \ .$$

When $E_i(1) > E_i(0)$ holds, then we can write

$$\operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) = \operatorname*{argmin}_{y_i \in \{0,1\}} E_i(y_i) - E_i(0) \ .$$



Graph construction: pairwise energy

Let us consider the pairwise energy function $E_{ij}(y_i, y_j) : \mathbb{B}^2 \to \mathbb{R}$. The possible values of $E_{ij}(y_i, y_j)$ are shown in the table:

E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

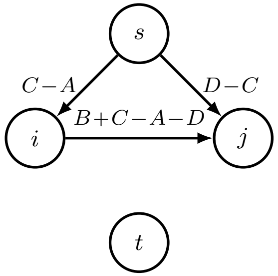
We furthermore assume that $E_{ij}(y_i, y_j)$ is regular, that is

$$
\begin{aligned}
E_{ij}(0,0) + E_{ij}(1,1) &\leqslant E_{ij}(0,1) + E_{ij}(1,0) \\
A + D &\leqslant B + C \,.
\end{aligned}
$$

Let us note that $E_{ij}(y_i, y_j)$ can be decomposed as:

$$
\begin{aligned}
\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline \end{array}
&= A + \begin{array}{|c|c|}\hline 0 & B-A \\ \hline C-A & D-A \\ \hline \end{array}
= A + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C-A & C-A \\ \hline \end{array}
+ \begin{array}{|c|c|}\hline 0 & B-A \\ \hline 0 & D-C \\ \hline \end{array} \\
&= A + \underbrace{\begin{array}{|c|c|}\hline 0 & 0 \\ \hline C-A & C-A \\ \hline \end{array}}_{E_i(1)}
+ \underbrace{\begin{array}{|c|c|}\hline 0 & D-C \\ \hline 0 & D-C \\ \hline \end{array}}_{E_j(1)}
+ \underbrace{\begin{array}{|c|c|}\hline 0 & B+C-A-D \\ \hline 0 & 0 \\ \hline \end{array}}_{B+C-A-D \geqslant 0}
\end{aligned}
$$

Graph construction: pairwise energy, $C - A \geqslant 0$, $D - C \geqslant 0$ *

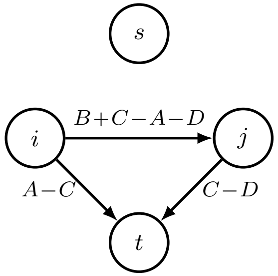


E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 0$.

$$C - A \geqslant 0 \ \Rightarrow C \geqslant A \ .$$
$$D - C \geqslant 0 \ \Rightarrow D \geqslant C \ \Rightarrow \ D \geqslant A \ .$$
$$0 \leqslant B + C - A - D \leqslant B - A \ \Rightarrow B \geqslant A \ .$$

Graph construction: pairwise energy, $A - C \geqslant 0$, $C - D \geqslant 0$ *

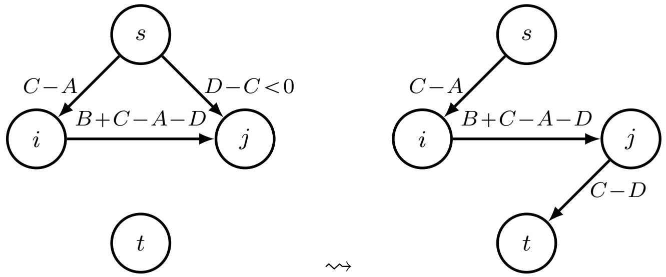


E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 1$.

$$\begin{aligned} C - D \geqslant 0 &\ \Rightarrow C \geqslant D\,. \\ A - C \geqslant 0 &\ \Rightarrow A \geqslant C \ \Rightarrow\ A \geqslant D\,. \\ 0 \leqslant B + C - A - D \leqslant B - D &\ \Rightarrow B \geqslant D\,. \end{aligned}$$

Graph construction: pairwise energy, $C - A \geqslant 0$**,** $C - D \geqslant 0$ *

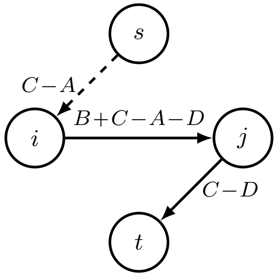


Note that the labeling $y_i = 1$, $y_j = 0$ is not possible in this case, since

$$C - A \geqslant 0 \ \Rightarrow \ C \geqslant A \; .$$

Graph construction: pairwise energy, $C - A \geqslant 0$, $C - D \geqslant 0$ *

Assume that $\min\{C - A, B + C - A - D, C - D\} = C - A$.



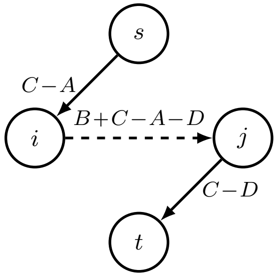
E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 1$.

$$C - A \leqslant B + C - A - D \;\Rightarrow 0 \leqslant B - C \;\Rightarrow\; B \geqslant D\,.$$
$$C - A \geqslant C - D \;\Rightarrow A \geqslant D\,.$$
$$C - D \geqslant 0 \;\Rightarrow C \geqslant D\,.$$

Graph construction: pairwise energy, $C - A \geqslant 0$, $C - D \geqslant 0$ *

Assume that $\min\{C-A, B+C-A-D, C-D\} = B+C-A-D$.



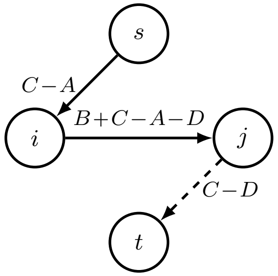
E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = 0$, $y_j = 1$.

$$B + C - A - D \leqslant C - A \;\Rightarrow B \leqslant D \,.$$
$$B + C - A - D \geqslant C - D \;\Rightarrow B \leqslant A \,.$$
$$C - A \geqslant 0 \;\Rightarrow A \leqslant C \;\Rightarrow\; B \leqslant C \,.$$

Graph construction: pairwise energy, $C - A \geqslant 0$, $C - D \geqslant 0$ *

Assume that $\min\{C-A, B+C-A-D, C-D\} = C-D$.

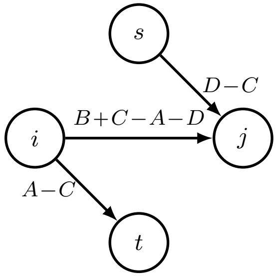


E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = y_j = 0$.

$$C - D \leqslant B + C - A - D \ \Rightarrow B \geqslant A\,.$$
$$C - D \leqslant C - A \ \Rightarrow D \geqslant A\,.$$
$$C - D \geqslant 0 \ \Rightarrow C \geqslant D \ \Rightarrow \ C \geqslant A\,.$$

Graph construction: pairwise energy, $A - C \geqslant 0$, $D - C \geqslant 0$ *



E_{ij}	$y_j = 0$	$y_j = 1$
$y_i = 0$	A	B
$y_i = 1$	C	D

Labeling: $y_i = 1$, $y_j = 0$.

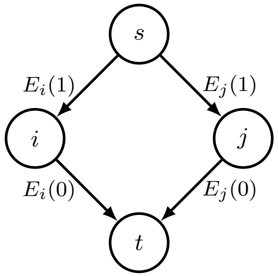
$$D - C \geqslant 0 \;\Rightarrow D \geqslant C \; .$$
$$A - C \geqslant 0 \;\Rightarrow A \geqslant C \; .$$
$$0 \leqslant B + C - A - D \leqslant B - A \;\Rightarrow B \geqslant A \;\Rightarrow\; B \geqslant C \; .$$

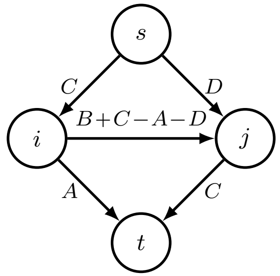
All the other cases can be similarly derived.

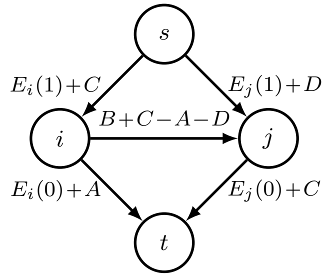
Graph construction

Putting all together we get that

Unaries	Pairwise	Overall energy
$\operatorname{argmin}_{\mathbf{y}} E_i(y_i) + E_j(y_j)$	$\operatorname{argmin}_{\mathbf{y}} E_{ij}(y_i, y_j)$	$\operatorname{argmin}_{\mathbf{y}} E_i(y_i) + E_j(y_j) + E_{ij}(y_i, y_j)$







Remarks

Regularity is an *extremely important* property as is allows to minimize energy functions by making use of graph cut. Moreover, without the regularity constraint, the problem become intractable.

Let E_2 be a nonregular function of two binary variables. Then minimizing the energy function

$$E(y_1, \dots, y_n) = \sum_i E_i(y_i) + \sum_{i<j} E_2(y_i, y_j) \ ,$$

where E_i are arbitrary functions of one binary variable, is NP-hard.

Multi-label problem

Multi-label problem

We define a label set $\mathcal{L} = \{1, 2, \dots, L\}$, where L is a (finite) constant. Therefore the output domain is defined as $\mathcal{Y} = \mathcal{L}^{\mathcal{V}}$. The *energy function* has the following form

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) \ ,$$

where $\mathbf{x}$ consists of an input image.

In order to ease to notation we will omit $\mathbf{x}$ and define the *energy function* simply as

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j) \ .$$

Metric $*$

A function $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}^+$ is called a **metric** if the following properties are satisfied:

1. **Identity of indiscernibles**: $d(\ell_1, \ell_2) = 0 \quad \Leftrightarrow \quad \ell_1 = \ell_2$ for all $\ell_1, \ell_2 \in \mathcal{L}$.
2. **Symmetry**: $d(\ell_1, \ell_2) = d(\ell_2, \ell_1)$ for all $\ell_1, \ell_2 \in \mathcal{L}$.
3. **Triangle inequality**: $d(\ell_1, \ell_3) \leqslant d(\ell_1, \ell_2) + d(\ell_2, \ell_3)$ for all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$.

Example: the **truncated absolute distance** $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x, y) = \min(K, |x - y|)$ is a *metric*, where K is some constant. (See Exercise)

If a function $d : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ satisfies the first two properties (i.e. identity of indiscernibles and symmetric), then it is called **semi-metric**.

Example: the **truncated quadratic function** $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(x, y) = \min(K, |x - y|^2)$ is a *semi-metric*, where K is some constant. (See Exercise)

$\alpha - \beta$ **swap**

$\alpha - \beta$ **swap** changes the variables that are labeled as $\ell \in \{\alpha, \beta\}$. Each of these variables can choose either α or β. We introduce the following notation

$$\mathcal{Z}_{\alpha\beta}(\mathbf{y}, \alpha, \beta) = \{\mathbf{z} \in \mathcal{Y} : z_i = y_i, \text{ if } y_i \notin \{\alpha, \beta\}, \text{ otherwise } z_i \in \{\alpha, \beta\}\} \, .$$

The minimization of the *energy function* E can be reformulated as follows:

$$\begin{aligned}
\mathbf{z}^* \in \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} E(\mathbf{z}) &= \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{i \in \mathcal{V}} E_i(z_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(z_i, z_j) \\
&= \operatorname*{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \Bigg[\underbrace{\sum_{i \in \mathcal{V},\, y_i \notin \{\alpha,\beta\}} E_i(y_i)}_{\text{constant}} + \underbrace{\sum_{i \in \mathcal{V},\, y_i \in \{\alpha,\beta\}} E_i(z_i)}_{\text{unary}} \\
&\quad + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \notin \{\alpha,\beta\}}} E_{ij}(y_i, y_j)}_{\text{constant}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \in \{\alpha,\beta\},\, y_j \notin \{\alpha,\beta\}}} E_{ij}(z_i, y_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i \notin \{\alpha,\beta\},\, y_j \in \{\alpha,\beta\}}} E_{ij}(y_i, z_j)}_{\text{unary}} + \underbrace{\sum_{\substack{(i,j) \in \mathcal{E} \\ y_i, y_j \in \{\alpha,\beta\}}} E_{ij}(z_i, z_j)}_{\text{pairwise}} \Bigg].
\end{aligned}$$

Local optimization

Let us consider $E_{ij}(z_i, z_j)$ for a given $(i,j) \in \mathcal{E}$:

E_{ij}	α	β
α	$E_{ij}(\alpha,\alpha)$	$E_{ij}(\alpha,\beta)$
β	$E_{ij}(\beta,\alpha)$	$E_{ij}(\beta,\beta)$

If we assume that $E_{ij} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ is a **semi-metric** for each $(i,j) \in \mathcal{E}$, then

$$E_{ij}(\alpha,\alpha) + E_{ij}(\beta,\beta) = 0 \leqslant E_{ij}(\alpha,\beta) + E_{ij}(\beta,\alpha) = 2E_{ij}(\alpha,\beta) \ ,$$

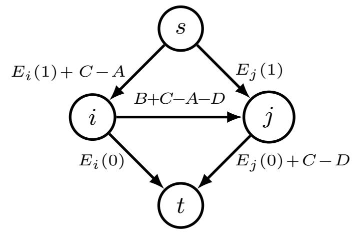
which means that E_{ij} is **regular** w.r.t. the labeling $\mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)$.

Graph construction for semi-metrics

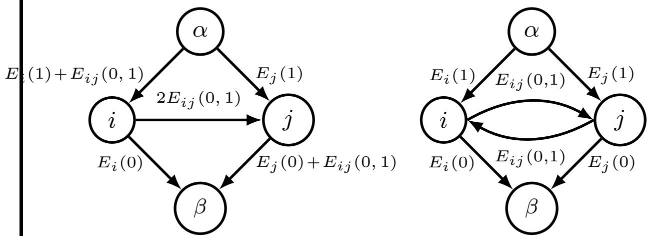
Let us consider the following *binary energy function*:

$$E(\mathbf{z}) = E_i(z_i) + E_j(z_j) + E_{ij}(z_i, z_j) \ ,$$

where E_{ij} is assumed to be a *semi-metric*.



Since E_{ij} is a *semi-metric*, we can construct a flow for $E(\mathbf{y})$ as follows:



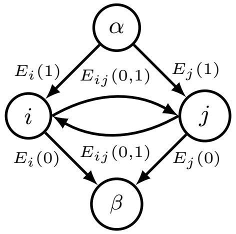
z_i	z_j	$E(\mathbf{z})$
0	0	$E_i(0)+E_j(0)$
0	1	$E_i(0)+E_j(1)+E_{ij}(1,0)$
1	0	$E_i(1)+E_j(0)+E_{ij}(1,0)$
1	1	$E_i(1)+E_j(1)$

Graph construction: t-links

We need to minimize the following **regular** *energy function*:

$$\mathbf{z}^* \in \operatorname*{argmin}_{\mathbf{z}\in\mathcal{Z}_{\alpha\beta}(\mathbf{y},\alpha,\beta)} \sum_{\substack{i\in\mathcal{V}\\ y_i\in\{\alpha,\beta\}}} E_i(z_i) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\in\{\alpha,\beta\},\, y_j\notin\{\alpha,\beta\}}} E_{ij}(z_i,y_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i\notin\{\alpha,\beta\},\, y_j\in\{\alpha,\beta\}}} E_{ij}(y_i,z_j) + \sum_{\substack{(i,j)\in\mathcal{E}\\ y_i,y_j\in\{\alpha,\beta\}}} E_{ij}(z_i,z_j).$$

Based on construction applied for *binary image segmentation*, we can also define a *flow network* $(\mathcal{V}',\mathcal{E}',c,\alpha,\beta)$, where $\mathcal{V}' = \{\alpha,\beta\}\cup\{i\in\mathcal{V} : y_i\in\{\alpha,\beta\}\}$ and $\mathcal{E}' = \underbrace{\{(\alpha,i),(i,\beta)\,|\,i\in\mathcal{V}'\backslash\{\alpha,\beta\}\}}_{\text{t-links}} \cup \underbrace{\{(i,j),(j,i)\,|\,i,j\in\mathcal{V}'\backslash\{\alpha,\beta\},(i,j)\in\mathcal{E}\}}_{\text{n-links}}$.

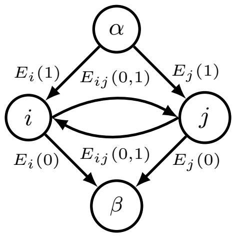


t-links: for all $i\in\mathcal{V}'\backslash\{\alpha,\beta\}$

$$c(\alpha,i) = E_i(\beta) + \sum_{(i,j)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ij}(\beta,y_j) + \sum_{(j,i)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ji}(y_j,\beta).$$

$$c(i,\beta) = E_i(\alpha) + \sum_{(i,j)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ij}(\alpha,y_j) + \sum_{(j,i)\in\mathcal{E},\, y_j\notin\{\alpha,\beta\}} E_{ji}(y_j,\alpha).$$

Graph construction: n-links



n-links: for all $(i,j)\in\mathcal{E}$, where $i,j\in\mathcal{V}'\backslash\{\alpha,\beta\}$

$$c(i,j) = c(j,i) = E_{ij}(\alpha,\beta)\ .$$

$\alpha - \beta$ swap algorithm *

Input: An *energy function* $E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j)$ to be minimized, where E_{ij} is a **semi-metric** for each $(i,j) \in \mathcal{E}$
Output: A local minimum $\mathbf{y} \in \mathcal{Y} = \mathcal{L}^{\mathcal{V}}$ of $E(\mathbf{y})$

1: Choose an arbitrary initial labeling $\mathbf{y} \in \mathcal{Y}$
2: $\mathbf{y}^* \leftarrow \mathbf{y}$
3: **for all** $(\alpha, \beta) \in \mathcal{L} \times \mathcal{L}$ **do**
4:     find $\mathbf{z}^* \in \operatorname{argmin}_{\mathbf{z} \in \mathcal{Z}_{\alpha\beta}(\mathbf{y}^*, \alpha, \beta)} E(\mathbf{z})$
5:     $\mathbf{y}^* \leftarrow \mathbf{z}^*$
6: **end for**
7: **if** $E(\mathbf{y}^*) < E(\mathbf{y})$ **then**
8:     $\mathbf{y} \leftarrow \mathbf{y}^*$
9:     Goto Step 2
10: **end if**

$\alpha - \beta$ swap algorithm is guaranteed to terminate in a finite number of cycles. This algorithm computes at least $|\mathcal{L}|^2$ graph cuts, which may take a lot of time, even for moderately large label spaces.

Summary *

- A binary energy function E consisting of up to pairwise functions is **regular**, if for each term E_{ij} for all $i < j$ satisfies

$$E_{ij}(0,0) + E_{ij}(1,1) \leqslant E_{ij}(0,1) + E_{ij}(1,0) \ .$$

- The *minimization of regular energy functions* can be achieved via *minCut-maxFlow*.
- The *multi-label problem* for a finite label set $\mathcal{L}$

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; \mathbf{x}) + \sum_{(i,j) \in \mathcal{E}} E_{ij}(y_i, y_j; \mathbf{x}) \ ,$$

  can be approximately solved by applying $\alpha - \beta$ *swap*, if E_{ij} is semi-metric.

In the **next lecture** we will learn about

- *α-expansion*: approximate solution for the *multi-label problem*, if E_{ij} is metric
- *FastPD algorithm*: linear programming relaxation for *multi-label problem*

Literature *

1. Yuri Boykov and Vladimir Kolmogorov. An experimental comparison of Min-cut/Max-flow algorithms for energy minimization in vision. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(9):1124–1137, September 2004
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4. Sebastian Nowozin and Christoph H. Lampert. Structured prediction and learning in computer vision. *Foundations and Trends in Computer Graphics and Vision*, 6(3–4), 2010