Weekly Exercise 3

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σ-algebra (7 Points)

Exercise 1 (σ**-algebra**, 1 point)**.** Let Ω be a countable set. Prove that $\mathcal{A} = \mathcal{P}(\Omega)$ is a σ-algebra over Ω.

Exercise 2 (σ**-algebra**, 4 points)**.**

a) Let $\mathcal{A}$ be a σ-algebra over Ω. Show that $\Omega \in \mathcal{A}$.

b) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, such that

$$A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A} \quad \text{and}$$
$$A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A} \, .$$

Show that

$$\emptyset \in \mathcal{A} \Leftrightarrow \mathcal{A} \neq \emptyset \Leftrightarrow \Omega \in \mathcal{A}.$$

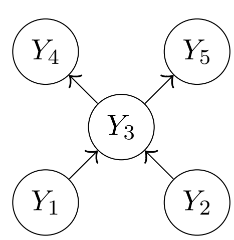
c) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra. Show that $\mathcal{A}$ is *closed under intersections*, i.e.

$$A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \in \mathcal{A} \, .$$

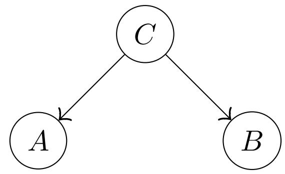
Exercise 3 (**Image measure**, 2 points)**.** Let $X : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ be a random variable and let P be a probability measure over $(\Omega, \mathcal{A})$. Show that the image measure P_X is a probability measure over $\mathcal{A}'$.

Graphical models (12 points)

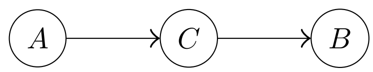
Exercise 4 (**Bayesian network**, 1 point)**.** Provide the factorization of $p(y_1, y_2, y_3, y_4, y_5)$ according to the following directed graphical model:



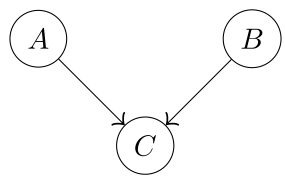
Exercise 5 (**Conditional independence in directed graphical models**, 6 point)**.** Without using the conditional independence property[1] of the Bayesian network, prove the following statements, while considering the following directed graphical models:



$A \perp\!\!\!\perp B \mid C$ and $A \not\!\perp\!\!\!\perp B \mid \emptyset$



$A \perp\!\!\!\perp B \mid C$ and $A \not\!\perp\!\!\!\perp B \mid \emptyset$



$A \not\!\perp\!\!\!\perp B \mid C$ and $A \perp\!\!\!\perp B \mid \emptyset$

Exercise 6 (**Proof of the Clifford-Hammersley theorem**, 2 points)**.** Assume an undirected graphical model $G = (\mathcal{V}, \mathcal{E})$ satisfying the local Markov property. Consider two non-connected nodes $a, b \in \mathcal{V}$, i.e. $(a, b) \notin \mathcal{E}$, and a subset of nodes $w \subset \mathcal{V}$ such that $a, b \notin w$. Show that

$$q(y_a \mid \mathbf{y}_w) \stackrel{\Delta}{=} p(y_a \mid \mathbf{y}_w, \mathbf{y}^*_{\mathcal{V}\setminus(w\cup\{a\})}) = q(y_a \mid y_b, \mathbf{y}_w) \ ,$$

where $p(\mathbf{y}_z, \mathbf{y}^*_{\bar{z}})$ is a joint probability.

Exercise 7 (**Factor graph**, 3 points)**.** Let G be a factor graph for a Markov random field consisting of N^2 binary variables, representing the pixels of an $N \times N$ image. For each pixel there is a unary potential, and there are pairwise potentials according to the 8-connected neighborhood.

a) Draw the factor graph for $N = 3$.

b) What is the total number of factors, depending on N, that are included in this model?

[1] In a Bayesian network a variable is conditionally independent of its non-descendants given its parents.