Probabilistic Graphical Models in Computer Vision (IN2329)

Csaba Domokos

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7. FastPD & Branch–and–MinCut

FastPD

Recall: Primal-dual LP for the multi-label problem *

The (relaxed) primal LP:

$$\min_{x_{i:\alpha}, x_{ij:\alpha\beta} \geqslant 0} \sum_{i \in \mathcal{V}} \sum_{\alpha \in \mathcal{L}} E_i(\alpha) x_{i:\alpha} + \sum_{(i,j) \in \mathcal{E}} w_{ij} \sum_{\alpha,\beta \in \mathcal{L}} d(\alpha,\beta) x_{ij:\alpha\beta}$$

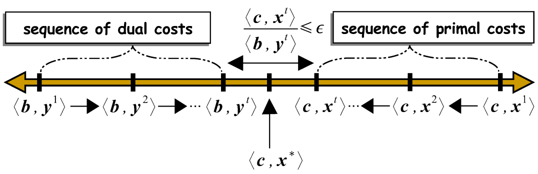
$$\begin{aligned} \text{subject to} \quad & \textstyle\sum_{\alpha \in \mathcal{L}} x_{i:\alpha} &= 1 \quad & \forall i \in \mathcal{V} \\ & \textstyle\sum_{\alpha \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{j:\beta} \quad & \forall \beta \in \mathcal{L}, (i,j) \in \mathcal{E} \\ & \textstyle\sum_{\beta \in \mathcal{L}} x_{ij:\alpha\beta} &= x_{i:\alpha} \quad & \forall \alpha \in \mathcal{L}, (i,j) \in \mathcal{E} \end{aligned}$$

The dual LP:

$$\max_{y_i, y_{ij:\alpha}, y_{ji:\beta}} \sum_{i \in \mathcal{V}} y_i$$

$$\begin{aligned} \text{subject to} \quad & y_i - \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:\alpha} &\leqslant E_i(\alpha) \quad & \forall i \in \mathcal{V}, \alpha \in \mathcal{L} \\ & y_{ij:\alpha} + y_{ji:\beta} &\leqslant w_{ij} d(\alpha,\beta) \quad & \forall (i,j) \in \mathcal{E}, \alpha,\beta \in \mathcal{L} \end{aligned}$$

Recall: Primal-dual schema *



Typically, primal-dual ϵ-approximation algorithms construct a sequence $(\mathbf{x}^k, \mathbf{y}^k)_{k=1,\ldots,t}$ of primal and dual solutions until the elements $\mathbf{x}^t$, $\mathbf{y}^t$ of the last pair are both **feasible** and **satisfy the relaxed primal complementary slackness conditions**, hence the condition $\langle \mathbf{c}, \mathbf{x} \rangle \leqslant \epsilon \langle \mathbf{b}, \mathbf{y} \rangle$ will be also fulfilled.

Pseudo-code of the FastPD algorithm *

1: $[\mathbf{x}\ \mathbf{y}] \leftarrow$ Init_Primals_Duals()
2: $labelChange \leftarrow$ false
3: **for all** $\alpha \in \mathcal{L}$ **do** ▷ α-iteration
4: $\quad \mathbf{y} \leftarrow$ PreEdit_Duals($\alpha, \mathbf{x}, \mathbf{y}$)
5: $\quad [\mathbf{x}'\ \mathbf{y}'] \leftarrow$ Update_Duals_Primals($\alpha, \mathbf{x}, \mathbf{y}$)
6: $\quad \mathbf{y}' \leftarrow$ PostEdit_Duals($\alpha, \mathbf{x}', \mathbf{y}'$)
7: $\quad$ **if** $\mathbf{x}' \neq \mathbf{x}$ **then**
8: $\qquad labelChange \leftarrow$ true
9: $\quad$ **end if**
10: $\quad \mathbf{x} \leftarrow \mathbf{x}'$ and $\mathbf{y} \leftarrow \mathbf{y}'$
11: **end for**
12: **if** $labelChange$ **then**
13: $\quad$ goto 2
14: **end if**
15: $\mathbf{y}^{\text{fit}} \leftarrow$ Dual_Fit($\mathbf{y}$)

PD1

Complementary slackness conditions *

From now on, in case of Algorithm PD1, we only assume that $d(\alpha,\beta) = 0 \Leftrightarrow \alpha = \beta$, and $d(\alpha,\beta) \geqslant 0$ (i.e. d is a semi-metric).

The *complementary slackness conditions* reduces to

$$y_i - \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:x_i} \geqslant \frac{E_i(x_i)}{\epsilon_1} \quad \Rightarrow \quad y_i \geqslant \frac{E_i(x_i)}{\epsilon_1} + \sum_{j \in \mathcal{V}:(i,j) \in \mathcal{E}} y_{ij:x_i}$$

$$y_{ij:x_i} + y_{ji:x_j} \geqslant \frac{w_{ij} d(x_i, x_j)}{\epsilon_2}$$

for specific values of $\epsilon_1, \epsilon_2 \geqslant 1$.

If $x_i = x_j = \alpha$ for neighboring pairs $(i,j) \in \mathcal{E}$, then

$$0 = w_{ij} d(\alpha,\alpha) \geqslant y_{ij:\alpha} + y_{ji:\alpha} \geqslant \frac{w_{ij} d(\alpha,\alpha)}{\epsilon_2} = 0 \,,$$

therefore we get that $y_{ij:\alpha} = -y_{ji:\alpha}$.

Complementary slackness conditions *

We have already known that $y_i = \min_{\alpha\in\mathcal{L}} h_i(\alpha)$. If $\epsilon_1 = 1$, then we get

$$y_i \geqslant \frac{E_i(x_i)}{\epsilon_1} + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:x_i} = h_i(x_i) \; .$$

Therefore

$$h_i(x_i) = \min_{\alpha\in\mathcal{L}} h_i(\alpha) \; , \tag{1}$$

which means that, at each vertex, **the active label should have the lowest height**.

If $\epsilon_2 = \epsilon_{\text{app}} := \frac{2d_{\max}}{d_{\min}}$, then the *complementary condition* simply reduces to:

$$y_{ij:x_i} + y_{ji:x_j} \geqslant \frac{w_{ij} d(x_i, x_j)}{\epsilon_{\text{app}}} \; . \tag{2}$$

It requires that any **two active labels should be raised proportionally to their "load"**.

Feasibility constraints *

To ensure feasibility of $\mathbf{y}$, PD1 enforces for any $\alpha\in\mathcal{L}$:

$$y_{ij:\alpha} \leqslant w_{ij} d_{\min}/2 \quad \text{where} \quad d_{\min} = \min_{\alpha\neq\beta} d(\alpha,\beta) \tag{3}$$

says that **there is an upper bound on how much we can raise a label**.

Hence, we get the feasibility condition

$$y_{ij:\alpha} + y_{ji:\beta} \leqslant 2w_{ij} d_{\min}/2 = w_{ij} d_{\min} \leqslant w_{ij} d(\alpha,\beta) \; .$$

Moreover the algorithm **keeps the active balance variables non-negative**, that is $y_{ij:x_i} \geqslant 0$ for all $i\in\mathcal{V}$.

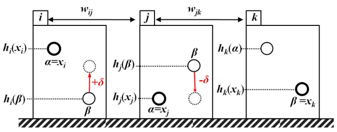
The *proportionality condition* (2) will be also fulfilled as $y_{ij:x_i}, y_{ji:x_j} \geqslant 0$ and if $y_{ij:x_i} = \frac{w_{ij} d_{\min}}{2}$, then

$$y_{ij:x_i} \geqslant \frac{w_{ij} d_{\min}}{2} \frac{d(x_i,x_j)}{d_{\max}} = \frac{w_{ij} d(x_i,x_j)}{\frac{2d_{\max}}{d_{\min}}} = \frac{w_{ij} d(x_i,x_j)}{\epsilon_{\text{app}}} \; .$$

Subroutine `Init_Primals_Duals()` *

1: $\mathbf{x}$ is simply initialized by a random label assignment ▷ Init primals
2: **for all** $(i,j)\in\mathcal{E}$ with $x_i \neq x_j$ **do** ▷ Init duals
3: $\quad y_{ij:x_i} \leftarrow w_{ij} d(x_i,x_j)/2$ and $y_{ji:x_i} \leftarrow -w_{ij} d(x_i,x_j)/2$
4: $\quad y_{ij:x_j} \leftarrow w_{ij} d(x_i,x_j)/2$ and $y_{ji:x_j} \leftarrow -w_{ij} d(x_i,x_j)/2$
5: **end for**
6: **for all** $i\in\mathcal{V}$ **do**
7: $\quad y_i \leftarrow \min_{\alpha\in\mathcal{L}} h_i(\alpha)$
8: **end for**
9: **return** $[\mathbf{x}\ \mathbf{y}]$

Update primal and dual variables



Dual variables update: Given the current active labels, any non-active label is raised, until it either reaches the active label, or attains the maximum raise allowed by the upper bound defined in (3).

Primal variables update: Given the new heights, there might still be vertices whose active labels are not at the lowest height. For each such vertex i, we select a non-active label, which is below x_i, but has already reached the maximum raise allowed by the upper bound defined in (3).

The optimal update of the α-heights can be simulated by pushing the **maximum amount of flow** through a flow network $G' = (\mathcal{V}\cup\{s,t\}, \mathcal{E}', c, s, t)$.

Graph construction: n-links

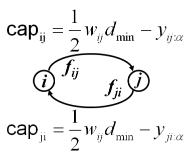
For each $(i,j)\in\mathcal{E}$, we insert two directed edges (i,j) and (j,i) into $\mathcal{E}'$.

The flow value f_{ij}, f_{ji} represent respectively the **increase, decrease of balance variable** $y_{ij:\alpha}$:

$$y'_{ij:\alpha} = y_{ij:\alpha} + f_{ij} - f_{ji} \quad \text{and} \quad y'_{ji:\alpha} = -y'_{ij:\alpha} \; .$$

According to (3), the capacities cap_{ij} and cap_{ji} are set based on

$$\text{cap}_{ij} + y_{ij:\alpha} = \frac{1}{2} w_{ij} d_{\min} = \text{cap}_{ji} + y_{ji:\alpha} \; .$$

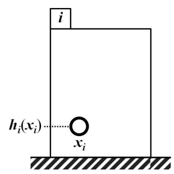


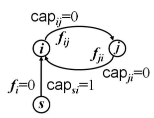
Graph construction: n-links

If α is already the active label of i (or j), then label α at i (or j) need not move.

Therefore, $y'_{ij:\alpha} = y_{ij:\alpha}$ and $y'_{ji:\alpha} = y_{ji:\alpha}$, that is

$$x_i = \alpha \text{ or } x_j = \alpha \quad \Rightarrow \quad \text{cap}_{ij} = \text{cap}_{ji} = 0 \; .$$





Graph construction: t-links *

Each node $i\in\mathcal{V}'\backslash\{s,t\}$ connects to either the source node s or the sink node t (but not to both of them).
There are three possible cases to consider:

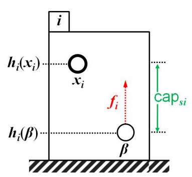
Case 1 ($h_i(\alpha) < h_i(x_i)$): we want to raise label α as much as it reaches label x_i.
We connect source node s to node i.
Due to the flow conservation property, $f_i = \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (f_{ij} - f_{ji})$ assuming the *more intuitive definition of flows* (see Lecture 4).
The flow f_i through that edge will then represent the total relative raise of label α:

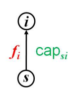
$$\begin{aligned} h_i(\alpha) + f_i &= \Big(E_i(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (f_{ij} - f_{ji}) \\ &= \Big(E_i(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y_{ij:\alpha}\Big) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} (y'_{ij:\alpha} - y_{ij:\alpha}) \\ &= E_i(\alpha) + \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} y'_{ij:\alpha} = h'_i(\alpha) \; . \end{aligned}$$

Graph construction: t-links

We need to raise up the ball corresponding to the label α only as high as the current active label of i, but not higher than that, we therefore set:

$$\text{cap}_{si} = h_i(x_i) - h_i(\alpha) \; .$$



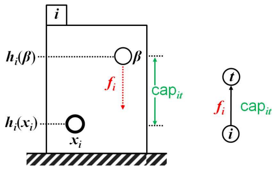


Case 2 ($h_i(\alpha) \geqslant h_i(x_i)$ **and** $c \neq x_i$): we can then afford a decrease in the height of α at i, as long as α remains above x_p.

We connect i to the sink node t through directed edge (i, t).

The flow f_i through edge it will equal the total relative decrease in the height of α:

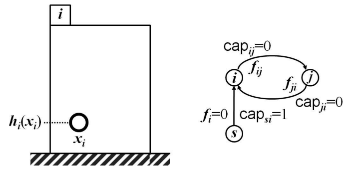
$$h_i'(\alpha) = h_i(\alpha) - f_i$$
$$\text{cap}_{it} = h_i(\alpha) - h_i(x_i)\ .$$



Case 3 ($\alpha = x_i$): we want to keep the height of α fixed at the current iteration.

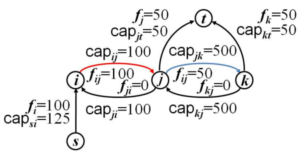
Note that the capacities of the *n-edges* for p are set to 0, since i has the *active label*. Therefore, $f_i = 0$ and $h'_{ij:\alpha} = h_{ij:\alpha}$.

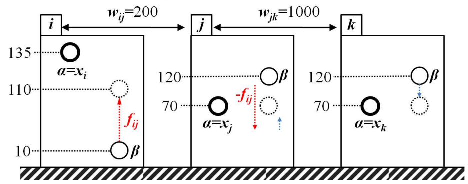
By convention $\text{cap}_{ij} := 1$.



Label α will be the new label of i (i.e. $x_i' = \alpha$) iff there exists *unsaturated path* (i.e. $f_{ij} < \text{cap}_{ij}$) between the source node s and node i. In all other cases, i keeps its current label (i.e. $x_i' = x_i$).

$$f_{ij} < \text{cap}_{ij}$$
$$h_i'(\alpha) - h_i(\alpha) < h_i(x_i) - h_i(\alpha)$$
$$h_i'(\alpha) < h_i(x_i) = h_i'(x_i)$$





1: $\mathbf{x}' \leftarrow \mathbf{x}$ and $\mathbf{y}' \leftarrow \mathbf{y}$
2: Apply max-flow to G' and compute flows f_i, f_{ij}
3: **for all** $(i,j) \in \mathcal{E}$ **do**
4: $\quad y'_{ij:\alpha} \leftarrow y_{ij:\alpha} + f_{ij} - f_{ji}$
5: **end for**
6: **for all** $i \in \mathcal{V}$ **do**
7: $\quad x_i \leftarrow \alpha \ \Leftrightarrow \exists$ unsaturated path $s \rightsquigarrow i$ in G'
8: **end for**
9: **return** $[\mathbf{x}'\ \mathbf{y}']$

The goal is to restore all *active balance variables* $y_{ij:x_i}$ to be non-negative.

1. $x_i' = \alpha \neq x_j'$: we have $\text{cap}_{ij}, y_{ij:\alpha} \geqslant 0$, therefore $y'_{ij:\alpha} = \text{cap}_{ij} + y_{ij:\alpha} \geqslant 0$.
2. $x_i' = x_j' = \alpha$: we have $y'_{ij:\alpha} = -y'_{ji:\alpha}$, therefore $\text{load}'_{ij} = y'_{ij:\alpha} + y'_{ji:\alpha} = 0$. By setting $y'_{ij}(\alpha) = y'_{ji:\alpha} = 0$ we get $\text{load}'_{ij} = 0$ as well.

Note that none of the "load" were altered.

1: **function** PostEdit_Duals$(\alpha,\mathbf{x}',\mathbf{y}')$
2: $\quad$**for all** $(i,j) \in \mathcal{E}$ with $(x_i' = x_j' = \alpha)$ and $(y'_{ij:\alpha} < 0$ or $y'_{ji:\alpha} < 0)$ **do**
3: $\quad\quad y'_{ij:\alpha} \leftarrow 0$ and $y'_{ji:\alpha} \leftarrow 0$
4: $\quad$**end for**
5: $\quad$**for all** $i \in \mathcal{V}$ **do**
6: $\quad\quad y_i' \leftarrow \min_{\alpha \in \mathcal{L}} h_i'(\alpha)$
7: $\quad$**end for**
8: $\quad$**return** $\mathbf{y}'$
9: **end function**

$\text{APF}^{\mathbf{x}',\mathbf{y}'} \leqslant \text{APF}^{\mathbf{x},\mathbf{y}}$, where $\text{APF}^{\mathbf{x},\mathbf{y}}$ is defined as

$$\begin{aligned}\text{APF}^{\mathbf{x},\mathbf{y}} \triangleq \sum_{i\in\mathcal{V}} h_i(x_i) &= \sum_{i\in\mathcal{V}} \Big(E_i(x_i) + \sum_{j\in\mathcal{V},(i,j)\in\mathcal{E}} \text{load}_{ij}^{\mathbf{x},\mathbf{y}}\Big) \\ &= \sum_{i\in\mathcal{V}} E_i(x_i) + \sum_{(i,j)\in\mathcal{E}} (y_{ij:x_i} + y_{ji:x_j}) \\ &\leqslant \sum_{i\in\mathcal{V}} E_i(x_i) + \sum_{(i,j)\in\mathcal{E}} w_{ij} d(x_i, x_j) = E(\mathbf{x})\ .\end{aligned}$$

This condition shows that the algorithm terminates (assuming integer capacities), due to the **reassign rule**, which ensures that a new *active label* has always lower *height* than the previous *active label*, i.e. $h_i'(x_i') \leqslant h_i(x_i)$.

In summary, one can see that `PD1` always leads to an ϵ-approximate solution:

Theorem 1. *The final primal-dual solutions generated by* `PD1` *satisfy*

1. $h_i(x_i) = \min_{\alpha\in\mathcal{L}} h_i(\alpha)$ *for all* $i \in \mathcal{V}$,
2. $x_i \neq x_j \Rightarrow \text{load}_{ij} \geqslant \frac{w_{ij} d(x_i,x_j)}{\epsilon_{app}}$ *for all* $(i,j) \in \mathcal{E}$,
3. $y_{ij:\alpha} \leqslant \frac{w_{ij} d_{\min}}{2}$ *for all* $(i,j) \in \mathcal{E}$ *and* $\alpha \in \mathcal{L}$,

and thus they satisfy the relaxed complementary slackness conditions with $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_{app} = \frac{2 d_{\max}}{d_{\min}}$.

PD2

Parameterization of the PD2 algorithm

We now assume that d is a *metric*.

In fact, PD2 represents a family of algorithms parameterized by $\mu \in [\frac{1}{\epsilon_{\text{app}}}, 1]$. Algorithm PD2$_\mu$ will achieve *complementary slackness conditions* with

$$\epsilon_1 \triangleq \mu\epsilon_{\text{app}} \geqslant \frac{1}{\epsilon_{\text{app}}}\epsilon_{\text{app}} \geqslant 1 \quad \text{and} \quad \epsilon_2 = \epsilon_{\text{app}} \; .$$

Algorithm PD1 always generates a feasible dual solution at any of its inner iterations, whereas PD2$_\mu$ **may allow any such dual solution to become infeasible**.

Dual-fitting: PD2$_\mu$ ensures that the (probably infeasible) final dual solution is "not too far away from feasibility", which practically means that if that solution is divided by a suitable factor, it will become feasible again.

Complementary slackness conditions *

Similarly to Algorithm PD1, the following equalities will hold for $i \in \mathcal{V}$

$$y_i = \min_{\alpha \in \mathcal{L}} h_i(\alpha) = h_i(x_i) \triangleq E_i(x_i) + \sum_{i \in \mathcal{V},(i,j)\in\mathcal{E}} y_{ij:x_i} \; .$$

PD2$_\mu$ generates a series of intermediate pairs satisfying *complementary slackness conditions* for $\epsilon_1 \geqslant 1$ and $\epsilon_2 \geqslant \frac{1}{\mu} = \frac{1}{1/\epsilon_{\text{app}}} = \epsilon_{\text{app}}$:

$$\frac{E_i(x_i)}{\epsilon_1} + \sum_{i \in \mathcal{V},(i,j)\in\mathcal{E}} y_{ij:x_i} \leqslant E_i(x_i) + \sum_{i \in \mathcal{V},(i,j)\in\mathcal{E}} y_{ij:x_i} \triangleq h_i(x_i) = y_i \qquad \forall i \in \mathcal{V} \; .$$

$$\frac{w_{ij} d(x_i, x_j)}{\epsilon_2} \leqslant \mu \cdot w_{ij} \cdot d(x_i, x_j) = \text{load}_{ij}^{\mathbf{x},\mathbf{y}} \qquad \forall (i,j) \in \mathcal{E} \; .$$

Like PD1, PD2$_\mu$ also maintains non-negativity of *active balance variables*.

Dual fitting

Instead of the *feasibility conditions* $y_{ij:\alpha} + y_{ji:\beta} \leqslant w_{ij} d(\alpha, \beta)$, PD2$_\mu$ maintains the conditions

$$y_{ij:\alpha} + y_{ji:\beta} \leqslant 2\mu w_{ij} d_{\max} \qquad \forall (i,j) \in \mathcal{E}, \; \forall \alpha, \beta \in \mathcal{L} \; .$$

Therefore, the *dual solution* of the last intermediate pair may be *infeasible*. However, these conditions ensure that the last dual solution $\mathbf{y}$, is not "too far away from feasibility". By replacing $\mathbf{y}$ with $\mathbf{y}^{\text{fit}} = \frac{\mathbf{y}}{\mu\epsilon_{\text{app}}}$ we get that

$$y_{ij:\alpha}^{\text{fit}} + y_{ji:\beta}^{\text{fit}} = \frac{y_{ij:\alpha} + y_{ji:\beta}}{\mu\epsilon_{\text{app}}} \leqslant \frac{2\mu w_{ij} d_{\max}}{\mu\epsilon_{\text{app}}} = \frac{2\mu w_{ij} d_{\max}}{\mu 2 d_{\max}/d_{\min}} = w_{ij} d_{\min} \leqslant w_{ij} d(\alpha, \beta).$$

This means that $\mathbf{y}^{\text{fit}}$ is **feasible**.

function DUAL_FIT($\mathbf{y}$)
  return $\mathbf{y}^{\text{fit}} \leftarrow \frac{\mathbf{y}}{\mu\epsilon_{\text{app}}}$
end function

Update primal and dual variables

The main/only difference in the subroutine Update_Duals_Primals(α,$\mathbf{x}$,$\mathbf{y}$) is the definition of the capacities corresponding to the *n-edges*. More precisely, assuming an α-iteration, where $x_i = \beta \neq \alpha$ and $x_j = \gamma \neq \alpha$ for a given $(i,j) \in \mathcal{E}$:

$$\text{cap}_{ij} = \mu w_{ij}(d(\beta, \alpha) + d(\alpha, \gamma) - d(\beta, \gamma)) \geqslant 0 \; , \qquad (4)$$

$$\text{cap}_{ji} = 0 \; .$$

All the capacities in the flow network **must** be non-negative. This motivates that d must be a metric.

By applying $\text{load}_{ij}^{\mathbf{x},\mathbf{y}} = y_{ij:\beta} + y_{ji:\gamma} = \mu w_{ij} d(\beta, \gamma)$ one can get

$$\begin{aligned} y'_{ij:\alpha} &= y_{ij:\alpha} + \text{cap}_{ij} = y_{ij:\alpha} + \mu w_{ij}(d(\beta, \alpha) + d(\alpha, \gamma) - d(\beta, \gamma)) \\ &= y_{ij:\alpha} + (y_{ij:\beta} + y_{ji:\alpha}) + \mu w_{ij} d(\alpha, \gamma) - (y_{ij:\beta} + y_{ji:\gamma}) = \mu w_{ij} d(\alpha, \gamma) - y_{ji:\gamma} \; , \end{aligned}$$

which ensures that

$$\text{load}_{ij}^{\mathbf{x},\mathbf{y}} = y_{ij:\alpha} + y_{ji:\gamma} = \big(\mu w_{ij} d(\alpha, \gamma) - y_{ji:\gamma}\big) + y_{ji:\gamma} = \mu w_{ij} d(\alpha, \gamma) \; .$$

Subroutine PreEdit_Duals(α,$\mathbf{x}$,$\mathbf{y}$) *

The role of this routine is to edit current solution $\mathbf{y}$, before the subroutine Update_Duals_Primals(α,$\mathbf{x}$), so that

$$\text{load}_{ij}^{\mathbf{x},\mathbf{y}} = y_{ij:\alpha} + y_{ji:\gamma} = \mu w_{ij} d(\alpha, \gamma) \; .$$

function PREEDIT_DUALS(α,$\mathbf{x}$,$\mathbf{y}$)
  for all $(i,j) \in \mathcal{E}$ with $x_i \neq \alpha$, $x_j \neq \alpha$ **do**
    $y_{ij:\alpha} \leftarrow \mu w_{ij} d(\alpha, \gamma) - y_{ji:\gamma}$
    $y_{ji:\alpha} \leftarrow y_{ji:\gamma} - \mu w_{ij} d(\alpha, \gamma)$
  end for
  return $\mathbf{y}$
end function

Equivalence of PD2$_{\mu=1}$ and α-expansion

One can show that PD2$_{\mu=1}$ indeed generates an ϵ_{app} solution.

If $\mu = 1$, then $\text{load}_{ij}^{\mathbf{x},\mathbf{y}} = w_{ij} d(x_i, x_j)$. It can be shown that $\text{APF}^{\mathbf{x},\mathbf{y}} = E(\mathbf{x})$, whereas in any other case $\text{APF}^{\mathbf{x},\mathbf{y}} \leqslant E(\mathbf{x})$.
If $\mu < 1$, then the primal (dual) objective function necessarily decreases (increases) per iteration. Instead, APF constantly decreases.

Recall that APF is the sum of active labels' heights and PD2$_{\mu=1}$ always tries to choose the *lowest* label among x_i and α. During an α-iteration, PD2$_{\mu=1}$ chooses an $\mathbf{x}'$ that minimizes APF with respect to any other α-expansion $\bar{\mathbf{x}}$ of current solution $\mathbf{x}$.

Theorem 2. *Let $(\mathbf{x}', \mathbf{y}')$ denote the next primal-dual pair due to an α-iteration and let $\bar{\mathbf{x}}$ denote α-expansion of the current primal. Then*

$$E(\mathbf{x}') = APF^{\mathbf{x}',\mathbf{y}'} \leqslant APF^{\bar{\mathbf{x}},\mathbf{y}'} \leqslant E(\bar{\mathbf{x}}) \; .$$

$E(\mathbf{x}') \leqslant E(\bar{\mathbf{x}})$ shows that the α-expansion algorithm is equivalent to PD2$_{\mu=1}$.

Results: Stereo matching *

Original (left) PD1 PD2$_{\mu=1}$ with Potts

Remark: By modifying the Algorithm PD2$_{\mu=1}$, one could get Algorithm PD3, which can be applied even if d is a non-metric function.

Distance $d(\alpha,\beta)$	$\epsilon_{\text{app}}^{\text{PD1}}$	$\epsilon_{\text{app}}^{\text{PD2}_{\mu=1}}$	$\epsilon_{\text{app}}^{\text{PD3}_a}$	$\epsilon_{\text{app}}^{\text{PD3}_b}$	$\epsilon_{\text{app}}^{\text{PD3}_c}$	ϵ_{app}
$[\![\alpha \neq \beta]\!]$	1.0104	1.0058	1.0058	1.0058	1.0058	2
$\min(5, \lvert\alpha - \beta\rvert)$	1.0226	1.0104	1.0104	1.0104	1.0104	10
$\min(5, \lvert\alpha - \beta\rvert^2)$	1.0280	-	1.0143	1.0158	1.0183	10

Branch-and-MinCut

Introduction

We address the problem of **binary image segmentation**, where we also consider *non-local parameters* that are known *a priori*.

For example, one can assume prior knowledge about the **shape** of the foreground segment or the **color distribution** of the foreground and/or background.

Let us consider an *undirected graphical model* $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of pixels and $\mathcal{E}$ consists of 8-connected pairs of pixels. We define the energy function $E : \mathbb{B}^{\mathcal{V}} \times \Omega \to \mathbb{R}$ for non-local parameter $\omega \in \Omega$:

$$E(\mathbf{y}, \omega) = C(\omega) + \sum_{i \in \mathcal{V}} F^i(\omega) \cdot y_i + \sum_{i \in \mathcal{V}} B^i(\omega) \cdot (1 - y_i) + \sum_{(i,j) \in \mathcal{E}} P^{ij}(\omega) \cdot |y_i - y_j| \ ,$$

where $C(\omega)$ is a *constant energy* w.r.t. $\mathbf{y}$, and $F^i(\omega)$ and $B^i(\omega)$ are the *unary energies* defining the cost of assigning the pixel i to the foreground and to the background, respectively. $P^{ij}(\omega) \in \mathbb{R}_0^+$ is **non-negative** for each $(i,j) \in \mathcal{E}$ ensuring the tractability of $E(\mathbf{x}, \omega)$.

Globally optimal segmentation *

The segmentation is given by a binary labeling $\mathbf{y} \in \mathbb{B}^{\mathcal{V}} = \{0,1\}^{\mathcal{V}}$, where 1 and 0 denote the background and the foreground, respectively. We also assume that the *non-local parameter* $\omega \in \Omega$ are taken from a **discrete set**.

Shape priors can be encoded as a product space of various poses and deformations of the *template*, while color priors will correspond to the set of parametric color distributions.

The goal is to achieve a **globally optimal** segmentation under non-local priors. The applied optimization method relies on two techniques: **branch-and-bound** and **graph cuts**.

Although a global minimum can be achieved, the worst case complexity of the method is large (essentially, the same as the exhaustive search over the space of non-local parameters).

An alternative way to solve the problem is to apply *alternating minimization*.

Lower bound

Let $L(\Omega)$ denote the *lower bound* for $E(\mathbf{y}, \omega)$ over $\mathbb{B}^{\mathcal{V}} \times \Omega$:

$$\begin{aligned}
&\min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}, \omega \in \Omega} E(\mathbf{y}, \omega) \\
&= \min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}, \omega \in \Omega} \Big\{ C(\omega) + \sum_{i \in \mathcal{V}} F^i(\omega) \cdot y_i + \sum_{i \in \mathcal{V}} B^i(\omega) \cdot (1 - y_i) + \sum_{(i,j) \in \mathcal{E}} P^{ij}(\omega) \cdot |y_i - y_j| \Big\} \\
&\geqslant \min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} \Big\{ \min_{\omega \in \Omega} C(\omega) + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega} F^i(\omega) \cdot y_i + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega} B^i(\omega) \cdot (1 - y_i) + \\
&\quad \sum_{(i,j) \in \mathcal{E}} \min_{\omega \in \Omega} P^{ij}(\omega) \cdot |y_i - y_j| \Big\} \\
&= \min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} \Big\{ C_\Omega + \sum_{i \in \mathcal{V}} F^i_\Omega(\omega) \cdot y_i + \sum_{i \in \mathcal{V}} B^i_\Omega(\omega) \cdot (1 - y_i) + \sum_{(i,j) \in \mathcal{E}} P^{ij}_\Omega(\omega) \cdot |y_i - y_j| \Big\} \\
&= L(\Omega) \ .
\end{aligned}$$

C_Ω, F^i_Ω, B^i_Ω and P^{ij}_Ω denote the minima of $C(\omega)$, $F^i(\omega)$, $B^i(\omega)$ and $P^{ij}(\omega)$, respectively, over $\omega \in \Omega$ and they are referred to as **aggregated energies**.

Monotonicity

Suppose $\Omega_1 \subset \Omega_2$, then the inequality $L(\Omega_1) \geqslant L(\Omega_2)$ holds.

Proof. Let us define $A(\mathbf{y}, \Omega)$ as

$$\begin{aligned}
A(\mathbf{y}, \Omega) \triangleq \min_{\omega \in \Omega} C(\omega) + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega} F^i(\omega) \cdot y_i + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega} B^i(\omega) \cdot (1 - y_i) \\
+ \sum_{(i,j) \in \mathcal{E}} \min_{\omega \in \Omega} P^{ij}(\omega) \cdot |y_i - y_j| \ .
\end{aligned}$$

Assume $\Omega_1 \subset \Omega_2$. Then, for any $\mathbf{y} \in \mathbb{B}^{\mathcal{V}}$

$$\begin{aligned}
&A(\mathbf{y}, \Omega_1) \\
&= \min_{\omega \in \Omega_1} C(\omega) + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega_1} F^i(\omega) y_i + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega_1} B^i(\omega)(1 - y_i) + \sum_{(p,q) \in \mathcal{E}} \min_{\omega \in \Omega_1} P^{ij}(\omega)|y_i - y_j| \\
&\geqslant \min_{\omega \in \Omega_2} C(\omega) + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega_2} F^i(\omega) y_i + \sum_{i \in \mathcal{V}} \min_{\omega \in \Omega_2} B^i(\omega)(1 - y_i) + \sum_{(i,j) \in \mathcal{E}} \min_{\omega \in \Omega_2} P^{ij}(\omega)|y_i - y_j| \\
&= A(\mathbf{y}, \Omega_2) \ . \qquad (5)
\end{aligned}$$

Monotonicity

Proof. Continued

Note that $L(\Omega) \triangleq \min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} A(\mathbf{y}, \Omega)$.

Let $\mathbf{y}_1^* \in \operatorname{argmin}_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} A(\mathbf{y}, \Omega_1)$ and $\mathbf{y}_2^* \in \operatorname{argmin}_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} A(\mathbf{y}, \Omega_2)$, then due to the monotonicity for $A(\mathbf{y}, \Omega)$ with respect to Ω (5), we get that

$$L(\Omega_1) \triangleq A(\mathbf{y}_1^*, \Omega_1) \geqslant A(\mathbf{y}_1^*, \Omega_2) \geqslant A(\mathbf{y}_2^*, \Omega_2) \triangleq L(\Omega_2) \ .$$

□

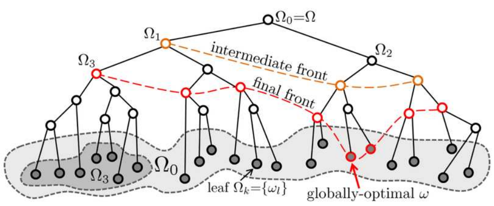
Computability and tightness

Computability: the lower bound $L(\Omega)$ equals the minimum of a *regular function*, which can be globally minimized via graph-cuts.

Tightness: for a singleton $\Omega = \{\omega\}$ (i.e. $|\Omega| = 1$) the bound $L(\Omega)$ is **tight**, that is

$$L(\{\omega\}) = \min_{\mathbf{y} \in \mathbb{B}^{\mathcal{V}}} E(\mathbf{y}, \omega) \ .$$

Best-first branch-and-bound optimization

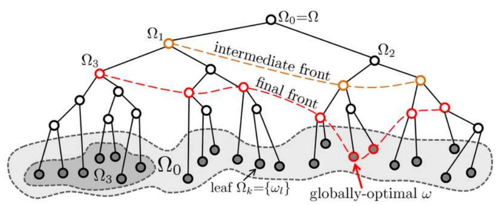


Source: Lempitsky et al.: Branch–and–MinCut: Global optimization for image segmentation with high–level–priors. JMIV, 2012.

The discrete domain Ω can be hierarchically clustered and the binary tree of its subregions can be considered.

At each step the *active node* with the **smallest** lower bound is removed from the *active front*, while two of its *children* are added to the *active front* (due to *monotonicity property* they have higher or equal lower bounds).

Best-first branch-and-bound optimization



If the *active node* with the smallest lower bound turns out to be a **leaf** ω' and $\mathbf{y}'$ is the corresponding optimal segmentation, then $E(\mathbf{y}', \omega') = L(\omega')$ due to the *tightness property*. Consequently, $(\mathbf{y}', \omega')$ is a **global minimum**.

Remark that in worst-case any optimization has to search exhaustively over Ω.

Pseudo code of Branch-And-Mincut *

```
Front ← ∅                                                        ▷ initializing the priority queue
[C_0, {F_0^i}, {B_0^i}, {P_0^ij}] ←GetAggregEnergies(Ω_0)
LB_0 ←GetMaxFlowValue({F_0^i},{B_0^i},{P_0^ij})+C_0
Front.InsertWithPriority(Ω_0,−LB_0)
while true do                                                    ▷ advancing front
    Ω ← Front.PullHighestPriorityElement()
    if IsSingleton(Ω) then                                       ▷ global minimum found
        ω ← Ω
        [C, {F^i}, {B^i}, {P^ij}] ←GetAggregEnergies(ω)
        x ←FindMinimumViaMincut({F^i},{B^i},{P^ij})
        return (x, ω)
    end if
    [Ω_1, Ω_2] ←GetChildrenSubdomains(Ω)
    [C_1, {F_1^i}, {B_1^i}, {P_1^ij}] ←GetAggregEnergies(Ω_1)
    LB_1 ←GetMaxFlowValue({F_1^i},{B_1^i},{P_1^ij})+C_1
    Front.InsertWithPriority(Ω_1,−LB_1)
    [C_2, {F_2^i}, {B_2^i}, {P_2^ij}] ←GetAggregEnergies(Ω_2)
    LB_2 ←GetMaxFlowValue({F_2^i},{B_2^i},{P_2^ij})+C_2
    Front.InsertWithPriority(Ω_2,−LB_2)
end while
```


Segmentation with shape priors

The prior is defined by the set of exemplar binary segmentations $\{\mathbf{x}^\omega \mid \omega \in \Omega\}$, where Ω is a discrete set indexing the exemplar segmentations.

We define a joint prior over the segmentation and the non-local parameter:

$$E_{\text{prior}}(\mathbf{y}, \omega) = \sum_{i \in \mathcal{V}} (1 - x_i^\omega) \cdot y_i + \sum_{i \in \mathcal{V}} x_i^\omega \cdot (1 - y_i) \,.$$

This encourages the segmentation $\mathbf{y}$ to be close in the Hamming-distance $(d_{\mathsf{H}}(\mathbf{a}, \mathbf{b}) = \frac{1}{N}\sum_{i=1}^N [\![a_i \neq b_i]\!])$ to one of the exemplar shapes.

The segmentation energy may be defined by adding a standard *contrast-sensitive Potts-model* for $\lambda, \sigma > 0$:

$$E(\mathbf{y}, \omega) = E_{\text{prior}}(\mathbf{y}, \omega) + \lambda \sum_{(i,j) \in \mathcal{E}} \frac{e^{-\frac{\|I_i - I_j\|}{\sigma}}}{|i - j|} \cdot |y_i - y_j| \,,$$

where I_i denotes RGB colors of the pixel i.

Parameterization: multiple templates × translations

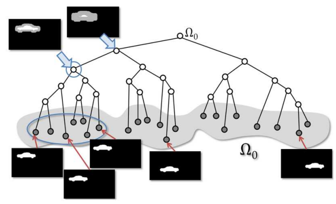
The shape prior is given by a set of templates, whereas each template can be located anywhere within the image.

$\Omega = \Delta \times \Theta$, where

- the set Δ indexes the set of all exemplar segmentations x_δ and
- Θ corresponds to translations.

Any exemplar segmentation $\mathbf{x}^\omega$ for $\omega = (\delta, \theta)$ is then defined as some *exemplar segmentation* x_δ centered at the origin and then *translated* by the shift θ.

Clustering tree



Source: Lempitsky et al.: Branch–and–MinCut: Global optimization for image segmentation with high–level–priors. JMIV, 2012.

For Δ we use *agglomerative bottom-up clustering* resulting in a (binary) clustering tree $T_\Delta = \{\Delta = \Delta_0, \Delta_1, \ldots, \Delta_N\}$.

To build a clustering tree for Θ, we recursively split along the "longer" dimension. This leads to a (binary) tree $T_\Theta = \{\Theta = \Theta_0, \Theta_1, \ldots, \Theta_N\}$.

Branch operation

Each *nodeset* Ω_t in the combined tree is defined by a pair $\Delta_t \times \Theta_t$.

The **looseness** of a nodeset Ω_t is defined as the number of pixels that change their mask value under different shapes in Ω_t (i.e. neither background nor foreground):

$$\Lambda(\Omega_t) = |\{i \mid \exists \omega_1, \omega_2 : x_i^{\omega_1} = 0 \text{ and } x_i^{\omega_2} = 1\}| \,.$$

The tree is built in a recursive top-down fashion as follows.

We start by creating a root nodeset $\Omega_0 = \Delta_0 \times \Theta_0$. Given a nodeset $\Omega_t = \Delta_t \times \Theta_t$ we consider (recursively) two possible splits:

1. split along the shape dimension or
2. split along the shift dimension.

The split that minimizes the sum of loosenesses is preferred.

The recursion stops when the leaf level is reached within both the *shape* and the *shift* trees.

Results *

Source: Lempitsky et al.: Branch–and–MinCut: Global optimization for image segmentation with high–level–priors. JMIV, 2012.

Yellow: global minimum of E **Blue**: feature-based car detector **Red**: global minimum of the combination of E with detection results (detection is included as a constant potential)

The prior set Δ was built by manual segmentation of 60 training images coming with the dataset.

Summary *

- Primal-dual schema for the *multi-labeling problem*:

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} E_i(y_i) + \sum_{(i,j) \in \mathcal{E}} w_{ij} \cdot d(y_i, y_j)$$

  - PD1: d is a semi-metric
  - PD2: d is a metric ($\text{PD2}_\mu = 1$ is equivalent to α-expansion)
  - PD3: d is a non-metric function (this case has not been discussed)
- For **binary image segmentation** we learned a global optimal solution, based on *branch and bound* optimization, when we are provided with (shape) prior information.

In the **next lecture** we will learn about **exact inference** (probabilistic and MAP) on tree structured factor graphs.

Literature *

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1. Victor Lempitsky, Andrew Blake, and Carsten Rother. Branch-and-Mincut: Global optimization for image segmentation with high-level priors. *Journal of Mathematical Imaging and Vision*, 44(3):315–329, March 2012