Weekly Exercise 2

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Probability theory (10 Points)

Exercise 1 (**Bayes' rule**, 1 point). Let A, B, C be *events*. Assuming $P(B \mid C) \neq 0$, prove that

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C) \cdot P(A \mid C)}{P(B \mid C)} .$$

Solution. By applying the definition of conditional probability, we get

$$\begin{aligned}\frac{P(B \mid A \cap C) \cdot P(A \mid C)}{P(B \mid C)} &= \frac{P(A \cap B \cap C) \cdot P(A \cap C)}{P(A \cap C) \cdot P(C)} \frac{P(C)}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \\ &= P(A \mid B \cap C) .\end{aligned}$$

Exercise 2 (**Bayes' rule**, 5 points). Siegfried the ornithologist does a study on the green-speckled swallow. Since he has a huge collection of bird photographs he wants to find all images depicting a green-speckled swallow. Due to it's distinctive features it is an easy task for Eduard, Siegfried's friend and computer vision scientist, to program a green-speckled swallow detector that marks all images containing such a bird. Unfortunately the detector does not work perfectly. If the image contains a green-speckled swallow the detector marks it correctly with a chance of 99.5%. If the image does not contain a green-speckled swallow the detector marks it correctly with a chance of 99.3%. The bird is also very rare: If we randomly draw an image from the collection, there is only a chance of 0.001% that the image contains a green-speckled swallow.

a) Do a formal modeling of the experiment. How does the discrete probability space look like?

b) What is the probability that a green-speckled swallow is on a given image, if the detector gives a positive answer?

c) What is the probability that a green-speckled swallow is on a given image, if the detector gives a negative answer?

Solution. a) The probability space $(\Omega, \mathcal{A}, P)$ is defined by

$$\begin{aligned}\Omega &= \{(s,+),(s,-),(n,+),(n,-)\}, \\ \mathcal{A} &= \{A \subset \Omega\}, \\ P &: \mathcal{A} \to [0,1],\end{aligned}$$

where the symbols $s, n, +, -$ have the following meaning:

$$\begin{array}{cl} s & \text{: image contains a green-speckled swallow} \\ n & \text{: image does not contain a green-speckled swalow} \\ + & \text{: detector reports a green-speckled swallow} \\ - & \text{: detector reports no green-speckled swallow} \end{array}$$

b) Let us introduce the following events

$$\begin{array}{rl} A = \{(s,+),(s,-)\} & \text{: the image contains a green-speckled swallow} \\ B = \{(s,+),(n,+)\} & \text{: the detector reports positive answer} \\ C = \{(s,-),(n,-)\} & \text{: the detector reports negative answer} \\ D = \{(n,+),(n,-)\} & \text{: the image does not contain green-speckled swallow} \end{array}$$

From the text we know that

$$P(B \mid A) = 0.995\ , \quad P(C \mid D) = 0.993 \quad \text{and} \quad P(A) = 0.00001\ .$$

Note that $\bar{A} = D$ and $\bar{B} = C$. Therefore

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A) \cdot P(A)}{P(B)} \\ &= \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid D) \cdot P(D)} \\ &= \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + (1 - P(C \mid D)) \cdot (1 - P(A))} = 1.41942 \cdot 10^{-3}\ . \end{aligned}$$

c)

$$\begin{aligned} P(A \mid C) &= \frac{P(C \mid A) \cdot P(A)}{P(C)} \\ &= \frac{(1 - P(B \mid A)) \cdot P(A)}{P(C \mid A) \cdot P(A) + P(C \mid D) \cdot P(D)} \\ &= \frac{(1 - P(B \mid A)) \cdot P(A)}{(1 - P(B \mid A)) \cdot P(A) + P(C \mid D) \cdot (1 - P(A))} = 5.03529 \cdot 10^{-8}\ . \end{aligned}$$

Exercise 3 (Independence, 2 points). Let A and B be events. Show that

$$P(A \mid B) = P(A) \quad \Leftrightarrow \quad P(A \cap B) = P(A) \cdot P(B)\ .$$

Solution. Let us start with the definition.

"$\Rightarrow$" Assume that $P(A \mid B) = P(A)$. Therefore,

$$\frac{P(A \cap B)}{P(B)} = P(A) \quad \Rightarrow \quad P(A \cap B) = P(A) \cdot P(B)\ .$$

"$\Leftarrow$" Assume that $P(A \cap B) = P(A)P(B)$. Therefore,

$$\frac{P(A \cap B)}{P(B)} = P(A) \quad \Rightarrow \quad P(A \mid B) = P(A)\ .$$

Exercise 4 (**Conditional independence**, 2 points)**.** Let A, B and C be events. Show that the two definitions of the conditional independence are equivalent, that is,

$$P(A \mid C) = P(A \mid B \cap C) \quad \Leftrightarrow \quad P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C) \ .$$

Solution. Let us start with the definition.

"⇒" Assume that $P(A \mid C) = P(A \mid B \cap C)$. Therefore,

$$\frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \quad \Rightarrow \quad P(A \cap B \cap C) = \frac{P(A \cap C) \cdot P(B \cap C)}{P(C)} \ .$$

We then have

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C) \cdot P(B \cap C)}{P(C)P(C)} = P(A \mid C) \cdot P(B \mid C) \ .$$

"⇐" Assume that $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$. Therefore,

$$\frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)} \frac{P(B \cap C)}{P(C)} \quad \Rightarrow \quad \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \ .$$

We then have

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} = P(A \mid B \cap C) \ ,$$