Weekly Exercise 6

Dr. Csaba Domokos
Technische Universität München, Computer Vision Group
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Metric (6 Points)

Exercise 1 (**Metric, semi-metric**, 6 Points). Show that the followings hold:

a) The *truncated absolute distance*, defined as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$

$$d(x, y) = \min(K, |x - y|), \quad \text{for some } K \in \mathbb{R}^+ ,$$

is a metric.

b) The *truncated quadratic function*, defined as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$

$$d(x, y) = \min(K, |x - y|^2), \quad \text{for some } K \in \mathbb{R}^+ ,$$

is a semi-metric.

c) The *weighted Potts-model*, defined as $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_0^+$

$$d(\ell_1, \ell_2) = w \cdot [\![\ell_1 \neq \ell_2]\!], \quad \text{for some } w \in \mathbb{R}^+ ,$$

is a metric.

Solution. a) Let $d(x, y) = \min(K, |x - y|)$ for some $K \in \mathbb{R}^+$. For all $x, y, z \in \mathbb{R}$

1)
$$d(x, y) = 0 \Leftrightarrow \min(K, 0) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x = y .$$

2)
$$d(x, y) = \min(K, |x - y|) = \min(K, |y - x|) = d(y, x) .$$

3) We will check the following cases:

- Assume that $K \leq |x - y|$ and $K \leq |y - z|$, then

$$d(x, y)+d(y, z) = \min(K, |x-y|)+\min(K, |y-z|) = K+K \geq \min(K, |x-z|) = d(x, z) .$$

- Assume that $K \leq |x - y|$ and $|y - z| < K$, then

$$d(x, y)+d(y, z) = \min(K, |x-y|)+\min(K, |y-z|) = K+|y-z| \geq \min(K, |x-z|) = d(x, z) .$$

- Assume that $|x - y| < K$ and $K \leq |y - z|$, then

$$d(x, y)+d(y, z) = \min(K, |x-y|)+\min(K, |y-z|) = |x-y|+K \geq \min(K, |x-z|) = d(x, z) .$$

- Assume that $|x-y| < K$ and $|y-z| < K$, then

$$\begin{aligned} d(x,y) + d(y,z) &= \min(K, |x-y|) + \min(K, |y-z|) \\ &= |x-y| + |y-z| \geq |x-z| \geq \min(K, |x-z|) = d(x,z) \ . \end{aligned}$$

Therefore $d(x,y)$ is a metric.

b) Let $d(x,y) = \min(K, |x-y|^2)$ for some $K \in \mathbb{R}^+$. For all $x, y \in \mathbb{R}$

1)
$$d(x,y) = 0 \Leftrightarrow \min(K, |x-y|^2) = 0 \Leftrightarrow |x-y|^2 = 0 \Leftrightarrow x = y \ .$$

2)
$$d(x,y) = \min(K, |x-y|^2) = \min(K, |y-x|^2) = d(y,x) \ .$$

Therefore, $d(x,y)$ is a semi-metric.

3) Let $d(\ell_1, \ell_2) = w \cdot [\![\ell_1 \neq \ell_2]\!]$ for some $w \in \mathbb{R}^+$. For all $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$

1)
$$d(\ell_1, \ell_2) = 0 \Leftrightarrow w \cdot [\![\ell_1 \neq \ell_2]\!] \Leftrightarrow \ell_1 = \ell_2 \ .$$

2)
$$d(\ell_1, \ell_2) = w \cdot [\![\ell_1 \neq \ell_2]\!] = w \cdot [\![\ell_2 \neq \ell_1]\!] = d(\ell_2, \ell_1) \ .$$

3) Assume that $\ell_1 = \ell_2 = \ell_3$, then

$$d(\ell_1, \ell_2) + d(\ell_2, \ell_3) = 0 = d(\ell_1, \ell_3) \ ,$$

otherwise

$$d(\ell_1, \ell_2) + d(\ell_2, \ell_3) \geq w \geq d(\ell_1, \ell_3) \ .$$

Therefore, $d(\ell_1, \ell_2)$ is a metric.

Programming (6 Points)

Exercise 2 (Binary image segmentation via maxFlow algorithm, 6 Points**).** Solve the *binary image segmentation* problem on the image in Figure 1 by applying the *Boykov–Kolmogorov maxFlow algorithm*.

For binary segmentation $y_i \in \mathbb{B}$ for all $i \in \mathcal{V}$, where $\mathcal{V}$ stands for the set of pixels, furthermore 0 and 1 denote the background and the foreground, respectively. Let us consider the following energy function for $w \in \mathbb{R}^+$:

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; x_i) + w \sum_{\{i,j\} \in \mathcal{E}} E_{ij}(y_i, y_j; x_i, x_j) \ , \quad (1)$$

where $\mathcal{E}$ includes 4-neighboring pixels and x_i consists of the RGB intensities of the pixel i.

Figure 1: The test image for binary image segmentation.

Use the GMM models f_{bg} and f_{fg} you trained in *Exercise 4* in order to define the **unary energy functions** for all $i \in \mathcal{V}$:

$$E_i(0; x_i) = -\log(f_{\text{bg}}(x_i)) ,$$
$$E_i(1; x_i) = -\log(f_{\text{fg}}(x_i)) .$$

Moreover, simply apply the *Potts-model* in order to define the **pairwise energy functions** for all $(i, j) \in \mathcal{E}$:

$$E_{ij}(y_i, y_j; x_i, x_j) = [\![y_i \neq y_j]\!] .$$

Construct a flow network corresponding to the defined energy function in Eq. (1) and solve the maximum flow problem by making use of the *Boykov–Kolmogorov algorithm*. You may use the provided maxFlow implementation of the algorithm found in the supplementary material `in2329-exercise_06_supp.zip`[1].

- Choose a set of different values for w, and report what you observe.
- How are the obtained segmentation results compared to the results you obtained in *Exercise 4* (i.e. without having pairwise terms, that is regularization)?

Minimum cut and maximum flow (4 Points)

Exercise 3 (Edmonds–Karp algorithm, 4 Points**).** Solve the maximum flow problem corresponding to the flow network in Figure 2 by applying the *Edmonds–Karp algorithm*. Find the minimum $s - t$ cut as well. Draw the residual network and the flow graph for each iteration.

[1] You may also find the implementation online http://pub.ist.ac.at/~vnk/software/maxflow-v3.04.src.zip

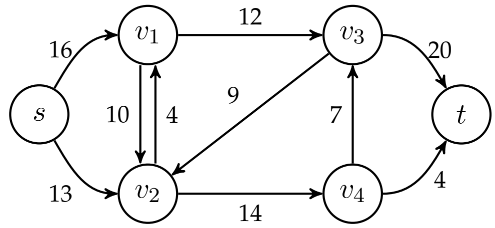

Figure 2: A flow network.

Solution. The maximum flow problem can be solved in three iterations using the *Edmonds–Karp algorithm*. The residual network and the flow for each iteration are shown below in Figure **??**.

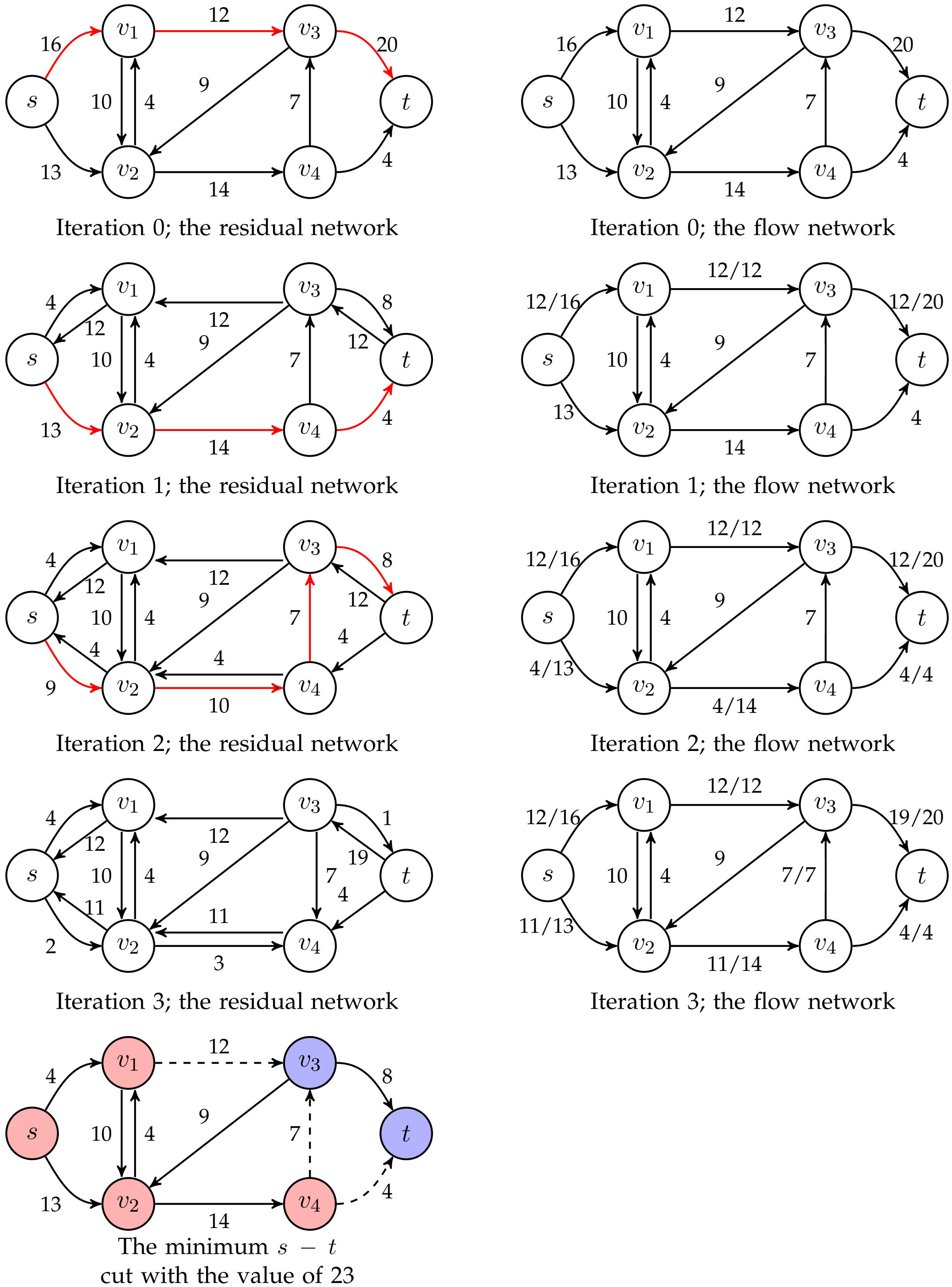

Figure 3: Solution for the maximum flow problem by applying the *Edmonds–Karp algorithm*. The shortest path that applied as an augmenting path in the next iteration is marked with red. The minimum cut is marked with dashed lines and the two partitions of the nodes are marked with red and blue, respectively.