Weekly Exercise 7

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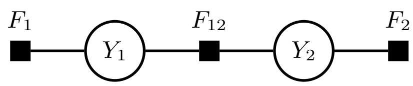
Fast Primal-Dual Schema (6 Points)

Exercise 1 (**Primal-dual LP**, 4 Points). Let us consider the following *factor graph* model with $\mathcal{Y}_1 = \{1,2,3\}$ and $\mathcal{Y}_2 = \{1,2\}$



where the factor energies are defined as follows:

$$E_1(y_1) = \begin{cases} 5\,, & \text{if } y_1 = 1 \\ 2\,, & \text{if } y_1 = 2 \\ 7\,, & \text{if } y_1 = 3 \end{cases}$$

$$E_2(y_2) = y_2$$

$E_{12}(y_1, y_2)$	1	2
1	0	1
2	1	0
3	4	1

Define the **primal** and the **dual** (relaxed) *linear programs*

$$\begin{aligned} &\min_{\mathbf{x}} \ \langle \mathbf{c}, \mathbf{x} \rangle \\ &\mathbf{A}\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} &\max_{\mathbf{y}} \ \langle \mathbf{b}, \mathbf{y} \rangle \\ &\mathbf{A}^T\mathbf{y} \leq \mathbf{c} \end{aligned}$$

for the *multi-labeling problem* corresponding to the factor graph above:

$$E(\mathbf{y}) = E_1(y_1) + E_2(y_2) + E_{12}(y_1, y_2)\ .$$

Solution. The indicator variables $\mathbf{x}$ with the corresponding costs $\mathbf{c}$ are given as

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}^T &= \begin{bmatrix} x_{1:1} & x_{1:2} & x_{1:3} & x_{2:1} & x_{2:2} & x_{12:11} & x_{12:12} & x_{12:21} & x_{12:22} & x_{12:31} & x_{12:32} \end{bmatrix}^T\ , \\ \mathbf{c} &= \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T \end{bmatrix}^T &= \begin{bmatrix} 5 & 2 & 7 & 1 & 2 & 0 & 1 & 1 & 0 & 4 & 1 \end{bmatrix}^T\ . \end{aligned}$$

The coefficient matrix $\mathbf{A}$ with the constant terms $\mathbf{b}$ for the constraints are given as

$$\mathbf{A} = \left(\begin{array}{ccccc|cccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

The dual variables are given as

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \mathbf{y}_3^T \end{bmatrix}^T = \begin{bmatrix} y_1 & y_2 & y_{12:1} & y_{12:2} & y_{21:1} & y_{21:2} & y_{21:3} \end{bmatrix}^T .$$

Exercise 2 (Complementary slackness, 2 Points)**.** Let $(\mathbf{x}, \mathbf{y})$ be a pair of *integral primal* and *dual feasible* solutions to the linear programming relaxation

$$\begin{aligned} &\min_{\mathbf{x}} \ \langle \mathbf{c}, \mathbf{x} \rangle && \max_{\mathbf{y}} \ \langle \mathbf{b}, \mathbf{y} \rangle \\ &\mathbf{A}\mathbf{x} = \mathbf{b} && \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \\ &\mathbf{x} \geq \mathbf{0} && \end{aligned}$$

corresponding to the multi-labeling problem. Show that if $(\mathbf{x}, \mathbf{y})$ satisfies the *relaxed primal complementary slackness conditions*, that is

$$\forall x_j > 0 \quad \Rightarrow \quad \sum_i a_{ij} y_i \geq \frac{c_j}{\varepsilon_j} ,$$

then $\mathbf{x}$ is an ε-approximation to the *optimal integral solution* $\mathbf{x}^*$ with $\varepsilon = \max_j \varepsilon_j$.

Solution. For all $x_j > 0$ assume that

$$\varepsilon_j (\mathbf{A}^T \mathbf{y})_j \geq c_j .$$

Therefore,

$$\langle \mathbf{c}, \mathbf{x} \rangle \leq \varepsilon \langle \mathbf{A}^T \mathbf{y}, \mathbf{x} \rangle = \varepsilon \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle \leq \varepsilon \langle \mathbf{y}, \mathbf{b} \rangle .$$

Thus for $\varepsilon \geq 1$, $(\mathbf{x}, \mathbf{y})$ is an ε-approximation to the optimal integral solution $\mathbf{x}^*$:

$$\langle \mathbf{c}, \mathbf{x}^* \rangle \leq \langle \mathbf{c}, \mathbf{x} \rangle \leq \varepsilon \langle \mathbf{b}, \mathbf{y} \rangle \leq \varepsilon \langle \mathbf{c}, \mathbf{x}^* \rangle .$$

Programming (6 Points)

Exercise 3 (Semantic image segmentation with α-expansion, 6 Points)**.** Consider the *energy function* for $w \geq 0$

$$E(\mathbf{y}; \mathbf{x}) = \sum_{i \in \mathcal{V}} E_i(y_i; x_i) + w \sum_{i,j \in \mathcal{E}} E_{ij}(y_i, y_j; x_i, x_j) \ , \tag{1}$$

for the *multi-labeling problem*, i.e. $\mathbf{y} \in \mathcal{L}^{\mathcal{V}}$, where $\mathcal{L}$ stands for the label set. $\mathcal{V}$ is the set of pixels of $\mathbf{x}$ and $\mathcal{E}$ consists of all four-neighboring pixels. Implement the α-expansion algorithm to solve **semantic image segmentation** for the images shown in figure 1. Try to choose different values for the parameter w for Equation (1) and compare the segmentation results.

Figure 1: The test images for semantic image segmentation.

These test images have been obtained from the MSRC image understanding dataset[1], which contains 21 classes, i.e. $\mathcal{L} = \{1, 2, \ldots, 21\}$. The meaning of the classes are given in the `21class.txt` file. Use it to check whether your results are reasonable.

To define the **unary energy functions** E_i, use the `*.c_unary` files provided in the supplementary material (`supp_07`). Each test image has its own unary file, specified by the same filename. From each unary file, you can read out a $K \times H \times W$ array of float numbers. The H and W are the image height and width, and $K = 21$ is the number of classes. This array contains the 21-class probability distribution for each pixel. You may find the `multilabel_demo.cpp` in the supplementary material, which demonstrates how to load a unary file and read out the corresponding probability values. The unary energy functions E_i for all $i \in \mathcal{V}$ are then defined as the *negative log-likelihood*.

The **pairwise energy functions** E_{ij} are defined by the *contrast sensitive Potts-model*

$$E_{ij}(y_i, y_j; x_i, x_j) = \exp(-\lambda \|x_i - x_j\|^2) [\![y_i \neq y_j]\!] \quad ,$$

where x_i is the intensity vector for pixel i, and you may choose $\lambda = 0.5$.

[1] https://www.microsoft.com/en-us/research/project/image-understanding/