Analysis of 3D Shapes (IN2238)

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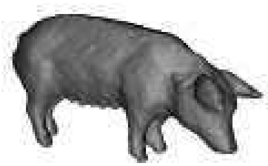
Matthias Vestner

Summer Semester 2017

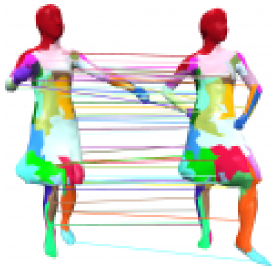
Computer Vision

We are always looking for master students!



3D Reconstruction

Optical Flow



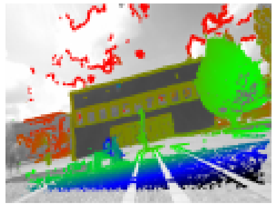
Shape Analysis

Robot Vision

RGB-D Vision

Image Segmentation

Convex Relaxation



Visual SLAM

Please talk to the appointed contact person directly.

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Schedule 4 / 26

IN2238

The lecture **Analysis of Three-Dimensional Shapes** will be organized as following:

- **Tuesday Lecture:** 10-11 and 11-12 in Room 02.09.023
- **Wednesday Tutorial:** 14-16 in Room 02.09.023
- **Thursday Lecture:** 14:00–14:45 and 15:00–15:45 in Room 02.09.023

The tutorial combines theoretical and programming assignments:

- **Assignment Distribution:** Tuesday 11:00-11:15 in Room 02.09.023
- **Theoretical Assignment Due:** Tuesday 23:59 per email
- **Assignment Presentation:** Wednesday 14-16 in Room 02.09.023

May

May 2017						
Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
1	2 Lecture 1	3	4 Lecture 2	5	6	7
8	9 Lecture 3	10 Matlab	11 Lecture 4	12	13	14
15	16 Lecture 5	17 Tutorial 1	18 Lecture 6	19	20	21
22	23 Lecture 7	24 Tutorial 2	25	26	27	28
29	30 Lecture 8	31 Tutorial 3				

June

June 2017						
Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
			1 Lecture 9	2	3	4
5	6	7 Tutorial 4	8 Lecture 10	9	10	11
12	13 Lecture 11	14 Tutorial 5	15	16	17	18
19	20 Lecture 12	21 Tutorial 6	22 Lecture 13	23	24	25
26	27 Lecture 14	28 Tutorial 7	29 Lecture 15	30		

July

July 2017						
Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
					1	2
3	4 Lecture 16	5 Tutorial 8	6 Lecture 17	7	8	9
10	11 Lecture 18	12 Tutorial 9	13 Lecture 19	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

Exams

Requirements for being admitted to the exam:

- **Registration:** Students need to be registered prior to the exam:
  May, 29^{th} – June, 30^{th} via TUM online.
- **Exam:** In the week of August, 14^{th} – 18^{th}.

Participation at the tutorial:

- **Not mandatory, but highly recommended:**
  Theoretical assignments will help to understand the topics of the lecture.
  Programming assignments will help to apply the theory to practical problems.
- **Bonus:** Active students who solve 60% of the assignments earn a bonus.
- **Exam:** If one receives a mark between 1.3 and 4.0 in the exam, the mark will be improved by 0.3 resp. 0.4. Marks of 1.0 or 5.0 cannot be improved.

Bonus

To achieve the bonus, the following requirements have to be fulfilled:

Theory

- 60% of all theoretical assignments have to be solved.
  (PDF-Submissions (using LaTeX) happen only online via email)
- At least one theoretical exercise has to be presented in front of the class.

Programming

- 60% of all programming assignments have to be presented
  during the tutorial.

To promote team work, we advocate to form groups of **two** or **three** students in order to solve and submit the assignments.

Team

Lecturers | **Teaching Assistant**

Dr. Frank R. Schmidt

Matthias Vestner

Zorah Lähner

Please do not hesitate to contact us in order to set up an appointment:

- f.schmidt@in.tum.de
- matthias.vestner@in.tum.de
- laehner@in.tum.de

Course Page

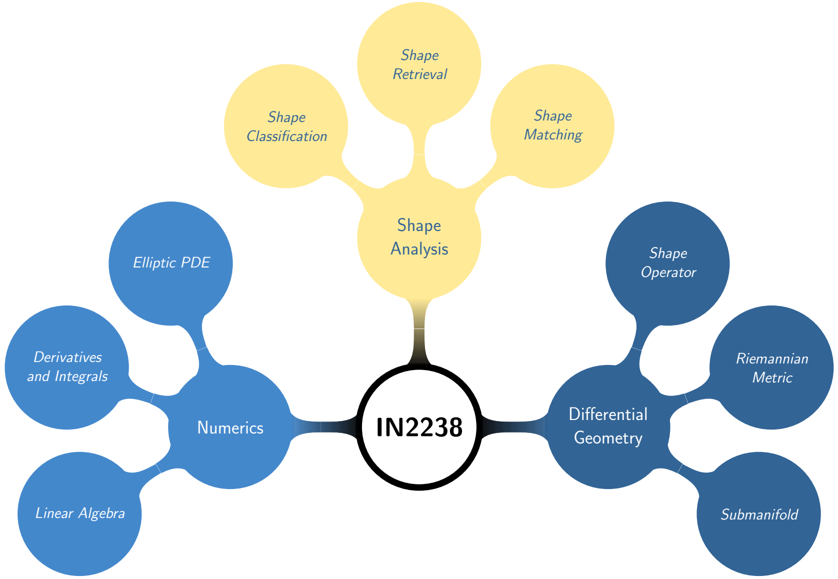
On the internal site of the course page you have access to extra course material:
`https://vision.in.tum.de/teaching/ss2017/shape_2238`

Password `Sh4p3`

- Slides for each lecture (Available prior to the lecture)
- Assignment Sheets (Available after the Tuesday lecture)
- Solution Sheets (Available after the Wednesday tutorial)

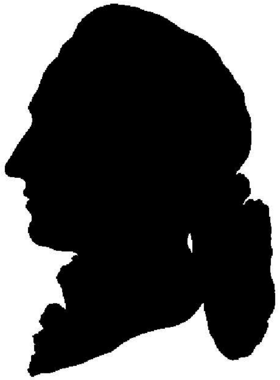
The course page will also be used for extra announcements.

Topics of the Lecture



What is a Shape?

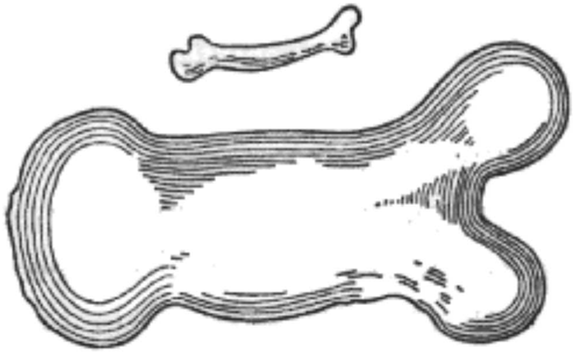
Cave Painting



Paper Cutting

- A **shape** describes the geometry of an object without relying on a specific location or orientation of this object, *i.e.*,
  An egg remains an egg, no matter where we put it on a table.
- A shape should reflect the human perception of objects:
  The whole is more than the sum of its parts.

What is a Shape?



Source: Galileo, 1638

Definition 1 (Kendall, 1977)**.** **Shape** is all the *geometrical information* that remains when *location, scale and rotational effects* are filtered out.

Two 3D-objects $O_1, O_2 \subset \mathbb{R}^3$ are **of the same shape** if we find a rotation $R \in \mathrm{SO}(3)$, a translation $T \in \mathbb{R}^3$ and a scaling factor $\sigma \in \mathbb{R}_+$ such that O_1 becomes O_2 after scaling, rotation and translation.

Shape Transformation

We can transform a 3D-object O_1 by applying a function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ to this object, *i.e.*,

$$\Phi(O_1) := \{\Phi(x) | x \in O_1\}$$

We write

$$\begin{aligned} O_1 + T :=& \Phi(O_1) & \Phi(x) =& x + T & \text{(Translation)} \\ \sigma \cdot O_1 :=& \Phi(O_1) & \Phi(x) =& \sigma \cdot x & \text{(Scaling)} \\ R \cdot O_1 :=& \Phi(O_1) & \Phi(x) =& R \cdot x & \text{(Rotation)} \end{aligned}$$

This means that O_2 is of the same shape as O_1 if the following is satisfied

$$O_2 = \sigma \cdot (R \cdot O_1 + T)$$

Shape as Equivalence Class of Objects

Note that the above mentioned relation between objects defines an **equivalence relation**:

$$O_1 \sim O_2 \quad :\Leftrightarrow \quad \left[\exists (R, T, \sigma) \in \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}_+ : O_2 = \sigma \cdot (R \cdot O_1 + T)\right]$$

One can easily show that the following holds:

$$\begin{aligned} O_1 &\sim O_1 & & & \text{(Reflexivity)} \\ O_1 &\sim O_2 & \Leftrightarrow \quad & O_2 \sim O_1 & \text{(Symmetry)} \\ O_1 \sim O_2, O_2 &\sim O_3 & \Rightarrow \quad & O_1 \sim O_3 & \text{(Transitivity)} \end{aligned}$$

Every equivalence relation defines an equivalence class of equivalent objects. The equivalence class $[O] := \{O' \subset \mathbb{R}^3 | O \sim O'\}$ of an object $O \subset \mathbb{R}^3$ is the **shape** of this object.

Restricted and Extended Shape Notions

In some applications, we like to differentiate between objects of different sizes. Analogously, we can define the **Shape-and-Scale equivalence relation**:

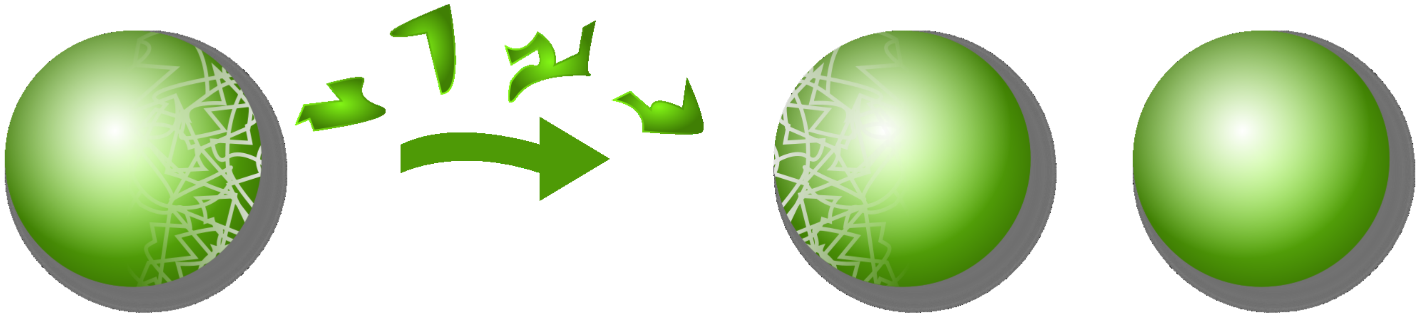
$$O_1 \sim O_2 \quad :\Leftrightarrow \quad \left[\exists (R,T) \in \mathrm{SO}(3) \times \mathbb{R}^3 : O_2 = R \cdot O_1 + T\right]$$

The equivalence class $[O] := \{O' \subset \mathbb{R}^3 | O \sim O'\}$ of an object $O \subset \mathbb{R}^3$ is the **shape-and-scale** of this object.

Some authors call also the shape-and-scale of an object their shape.

In some applications, we also like to extend the shape equivalence relation in order to be invariant with respect to different "poses",*i.e.*:
A horse is a horse, no matter whether it is standing or galopping.

Banach-Tarski Paradox



Source: Wikimedia.org

Theorem 1 (Banach and Tarski, 1924). *The 3D ball $B = \{x \in \mathbb{R}^3 | \langle x, x \rangle \leqslant 1\}$ can be partitioned into finite many sets A_i*, i.e., $B = \bigsqcup_{i=1}^{k} A_i$, *which can be rotated to form two copies of B.*

The proof of this paradox uses a heavy machinery based on set theory. In particular the so called *Axiom of Choice* is used to create the sets A_i.

Interestingly, we cannot measure the volume of any of the sets A_i. To avoid any similar problem, we are only interested in objects $O \subset \mathbb{R}^n$ that are open.

Open Sets

Definition 2 (Open Set)**.** A set $O \subset \mathbb{R}^n$ is called **open** iff

$$x \in O \quad \Rightarrow \quad \exists \varepsilon > 0 : B_\varepsilon(x) \subset O,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n | \, \|x - y\| < \varepsilon\}$ is a ball of radius ε centered at $x \in \mathbb{R}^n$.

Definition 3 (Relatively Open Set)**.** Given $X \subset \mathbb{R}^n$, we call $O \subset X$ **relatively open with respect to X** iff there exists an open set $\hat{O} \subset \mathbb{R}^n$ such that

$$O = \hat{O} \cap X$$

The set of all open sets is called its topology $\mathcal{T}(X)$.

Definition 4 (Neighborhood)**.** Given a subset $X \subset \mathbb{R}^n$ and $x \in X$, we call $O(x) \in \mathcal{T}(X)$ an **open neighborhood** of x if $x \in O$.

Boundary

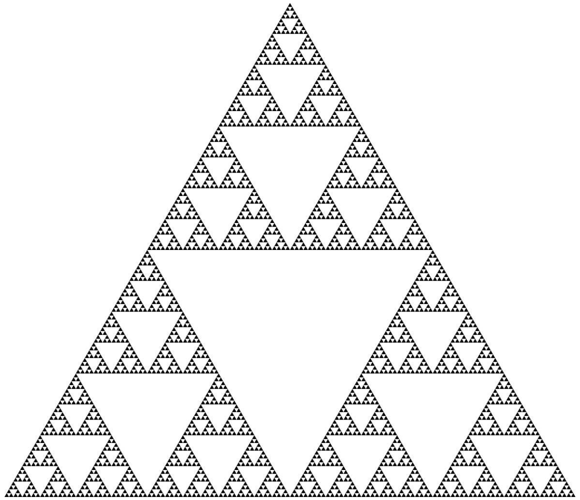
Definition 5 (Boundary)**.** Given a set $X \subset \mathbb{R}^n$, we call

$$\partial X := \{x \in \mathbb{R}^n | \forall O(x) \in \mathcal{T}(\mathbb{R}^n) : O \cap X \neq \emptyset \wedge O \backslash X \neq \emptyset\}$$

its **boundary**.

- The boundary of $\mathbb{R}^n$ is $\partial \mathbb{R}^n = \emptyset$.
- The boundary of the closed interval $[0, 1]$ is $\partial[0, 1] = \{0, 1\}$.
- The boundary of $[0, 1[$ is $\partial[0, 1[= \{0, 1\}$.
- The boundary of $]0, 1[$ is $\partial]0, 1[= \{0, 1\}$.
- The boundary of the ball $B^n = \{x \in \mathbb{R}^n | \langle x, x\rangle \leqslant 1\}$ is $\partial B = \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n | \langle x, x\rangle = 1\}$.

What is an Object?



Source: Wikimedia.org

To define what a shape is, we first need to define an **object**.

Definition 6 (Informal Definition)**.** An **object** of dimension d is an open subset $X \subset \mathbb{R}^d$ such that $\dim \partial X = d - 1$.

Note that we exclude sets whose boundaries are **fractals**. Instead, we want to have a boundary with the well-defined dimension $d - 1$.

Literature

Shapes

- Galileo, *Discorsi e dimostrazioni matematiche, informo a due nuove scienze attenti alla mecanica i movimenti locali*, 1638, appresso gli Elsivirii; Opere VIII. (2)
- Kendall, *The diffusion of shape*, 1977, Advances in Applied Probabilities (9), 428–430.
- Dryden and Mardia, *Statistical Shape Analysis*, 1998, Wiley, 376 pages.

Set Theoretical Results

- Banach and Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, 1924, Fundamenta Mathematica (6), 244–277.
- Carathéodory, *Vorlesungen über reelle Funktionen*, 1918, B. G. Teubner, 704 pages.