Analysis of 3D Shapes (IN2238)

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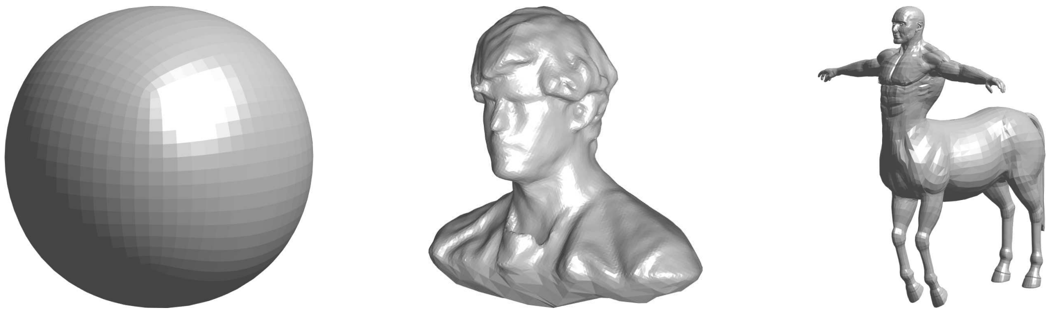
Linear Spaces 3 / 24

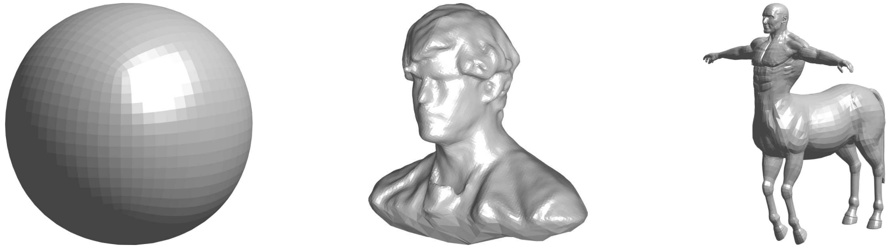
Shapes and Objects



The concept of a shape can be understood as a generalization of objects. In fact, we defined the **shape of an object** as a class of equivalent objects.

As a 3D-object we understand something like a ball or a human that occupies a certain region $X \subset \mathbb{R}^3$ of the real world.

Linear Algebra and Analysis



We defined an object of dimension d as an open subset $X \subset \mathbb{R}^d$ such that $\dim X = d - 1$. This concept of a dimension is in fact an extension of the dimension for linear vector spaces.

As a matter of fact, we like this boundary to be smooth.

Therefore, we will recap some of the main concepts from
Linear Algebra as well as **Analysis**.

Vector Spaces

An $\mathbb{R}$-vector space V is formally defined as an **Abelian group** that has some additional **linear** properties.

As an Abelian group, V possesses a binary operation $+$, a neutral element $0 \in V$ as well as an inverse element $(-v) \in V$ for each element $v \in V$.

In addition, there exists a scalar multiplication $\cdot$ such that

$$
\begin{aligned}
(\lambda \cdot \mu) \cdot v &= \lambda \cdot (\mu \cdot v) && \forall \lambda, \mu \in \mathbb{R}, v \in V \\
1 \cdot v &= v && \forall v \in V \\
(\lambda + \mu) \cdot v &= \lambda \cdot v + \mu \cdot v && \forall \lambda, \mu \in \mathbb{R}, v \in V \\
\lambda \cdot (u + v) &= \lambda \cdot u + \lambda \cdot v && \forall \lambda \in \mathbb{R}, u, v \in V
\end{aligned}
$$

Examples for vector spaces are $\mathbb{R}^n$, but also **function spaces** like $C^k(\mathbb{R}^n)$.

Vector Sub-Spaces

Given an $\mathbb{R}$-vector space V, the subset $U \subset V$ is also a vector space (called **subspace**) if the following holds:

$$\begin{aligned} u + v &\in U \qquad && \forall u, v \in U \\ \lambda v &\in U \qquad && \forall \lambda \in \mathbb{R}, v \in U \end{aligned}$$

Given a subset $X = \{x_1, \ldots, x_n\} \subset V$, the subset $\text{span}(X)$ is a subspace with

$$\text{span}(X) := \left\{ \sum_{i=1}^{n} \lambda_i x_i \middle| \lambda \in \mathbb{R}^n \right\}.$$

If the x_i are linear independent, we call X a **base** of $\text{span}(X)$
and n is called its **dimension**.

Linear Mapping

Given the $\mathbb{R}$-vector spaces U and V, a mapping $L\colon U \to V$ is a **linear mapping** if the following holds:

$$\begin{aligned} L(u+v) &= L(u) + L(v) && \forall u, v \in U \\ L(\lambda u) &= \lambda L(u) && \forall \lambda \in \mathbb{R}, u \in U \end{aligned}$$

Is X a basis of the n-dimensional vector space U and Y a basis of the m-dimensional vector space V, we obtain

$$L(x_j) = \sum_{i=1}^{m} a_{ij} y_i$$

$A \in \mathbb{R}^{m \times n}$ is then called the **representing matrix** of L with respect to the bases X and Y and we write:

$$\mathcal{M}_Y^X(L) = A.$$

Matrix-Multiplication

Given matrices $A \in \mathbb{R}^{m \times r}$ and $B \in \mathbb{R}^{r \times n}$, the product $C := A \cdot B \in \mathbb{R}^{m \times n}$ is

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{ir}} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mr} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & \mathbf{b_{1j}} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{r1} & \cdots & \mathbf{b_{rj}} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \mathbf{c_{ij}} & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}$$

with

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

It turns out that

$$\mathcal{M}_Z^Y(L_2) \cdot \mathcal{M}_Y^X(L_1) = \mathcal{M}_Z^X(L_2 \circ L_1)$$

Change of Bases

If we have $V = \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, the matrix-vector multiplication defines a linear mapping:

$$L\colon \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

Let us assume that we want to **change the bases** of $\mathbb{R}^n$. To that end, both X and Y can be written in matrix form and we have

$$\mathcal{M}_Y^X(L) = Y \cdot A \cdot X^{-1}$$

Thus, there is a subtle difference between linear mappings L and matrices A.
A is a representation of L that also takes the specific bases into account.

We say that two matrices A and B are **similar**, if there exists an invertible matrix X such that $B = X \cdot A \cdot X^{-1}$.

History of Differential

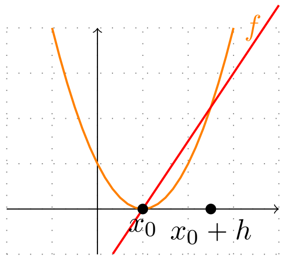
While the concept of the **derivative** or **differential** is nowadays one of the basic concepts in modern mathematics, it took a while to find a clean mathematical definition.

The notation $\frac{\mathrm{dy}}{\mathrm{dx}}$ is due to **Leibniz** who called **dx** and **dy** an "infinitely small change of" x resp. y.

In 1924, **Courant** mentioned that the idea of the differential as infinite small expression "lacks any meaning" and is therefore "useless".

The modern notion of derivatives and differential is due to **Cauchy** and **Weierstraß**, which we want to revise in the following.

Derivative according to Cauchy

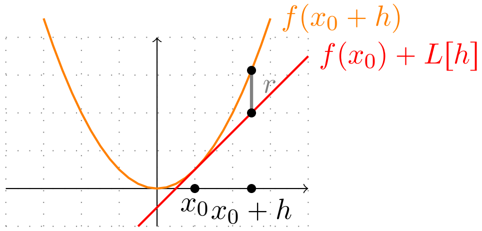


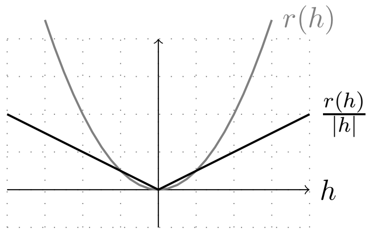
The derivative $f'(x_0)$ of a function $f \colon \mathbb{R} \to \mathbb{R}$ at the position $x_0 \in \mathbb{R}$ is

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

While this is a working mathematical definition, it is a bit difficult to extend it to arbitrary functions $f \colon \mathbb{R}^n \to \mathbb{R}^m$, since we cannot "divide by vectors".

Differential according to Weierstraß





Given a function $f\colon \mathbb{R} \to \mathbb{R}$ and a position $x_0 \in \mathbb{R}$, its **differential** $Df(x_0)$ is the **unique linear mapping** $L\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x_0 + h) = f(x_0) + L[h] + r(h)$$

$$\lim_{h \to 0} \frac{r(h)}{|h|} = 0$$

Jacobi Matrix

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function and $x_0 \in \mathbb{R}^m$. The differential

$$Df(x_0)\colon \mathbb{R}^m \to \mathbb{R}^n$$

is a linear mapping.

Using the canonical bases $\{e_1, \dots, e_m\}$ for $\mathbb{R}^m$ and $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$, $Df(x_0)$ can be written in matrix form, the **Jacobi matrix**

$$Df(x_0)[h] = J \cdot h \qquad J = \begin{pmatrix} J_{1,1} & \cdots & J_{1,m} \\ \vdots & & \vdots \\ J_{n,1} & \cdots & J_{n,m} \end{pmatrix}$$

with

$$J_{i,j} = \langle e_i, J \cdot e_j \rangle = \langle e_i, Df(x_0)[e_j] \rangle = \lim_{h \to 0} \frac{f^i(x_0 + h \cdot e_j) - f^i(x_0)}{h} = \partial_j f^i(x_0)$$

Chain Rule

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^k \to \mathbb{R}^m$ be differentiable functions. Then we have

$$\begin{aligned}(f \circ g)(x_0 + h) =& f\left(g(x_0) + Dg(x_0)[h] + r_g(h)\right) \\ =& (f \circ g)(x_0) + Df(g(x_0))\left[Dg(x_0)[h] + r_g(h)\right] + \\ & r_f\left(Dg(x_0)[h] + r_g(h)\right) \\ =& (f \circ g)(x_0) + Df(g(x_0))\left[Dg(x_0)[h]\right] + r(h)\end{aligned}$$

Thus we have

$$D(\mathbf{f} \circ \mathbf{g})(x_0) = \mathbf{Df}(g(x_0)) \cdot \mathbf{Dg}(x_0)$$

Chain Rule (Example)

Let $g_1, g_2 \colon \mathbb{R}^m \to \mathbb{R}^n$ and $f \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(x, y) := x + y$, we have

$$\begin{aligned} D(g_1 + g_2)(x_0) =& D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\ =& \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \end{pmatrix} \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} = Dg_1(x_0) + Dg_2(x_0) \end{aligned}$$

Let $g_1, g_2 \colon \mathbb{R}^m \to \mathbb{R}$ and $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $f(x, y) := x \cdot y$, we have

$$\begin{aligned} D(g_1 \cdot g_2)(x_0) =& D(f \circ g)(x_0) = Df(g(x_0)) \cdot Dg(x_0) \\ =& \begin{pmatrix} g_2(x_0) & g_1(x_0) \end{pmatrix} \cdot \begin{pmatrix} Dg_1(x_0) \\ Dg_2(x_0) \end{pmatrix} \\ =& Dg_1(x_0) \cdot g_2(x_0) + Dg_2(x_0) \cdot g_1(x_0) \end{aligned}$$

Inverse Function Theorem

If $f, f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are both continuously differentiable, we have

$$\begin{aligned} \mathrm{Id} =& D\left[f \circ f^{-1}(x)\right] = Df(f^{-1}(x)) \cdot D\left[f^{-1}\right](x) \\ D[f^{-1}](x) =& Df(f^{-1}(x))^{-1} \end{aligned}$$

Interestingly, also the opposite is (locally) true

Theorem 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ cont. differentiable and $Df(x_0)$ invertible. Then there exist neighborhoods $U(x_0)$ and $V(y_0)$ with $y_0 = f(x_0)$ such that*

- *$f\colon U \to V$ is a bijection.*
- *f is continuously differentiable.*
- *f^{-1} is continuously differentiable.*

Inverse Functions as Implicit Functions

Usually, one defines the **square root function** in an implicit manner:

$$x - \sqrt{x}^2 = 0$$

This can be formally done in the following way:

Theorem 2 (Square Root). *Let* $\Phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be the function*

$$\Phi(x, y) = x - y^2,$$

which satisfies $\Phi(x_0, y_0) = 0$ *for* $(x_0, y_0) = (4, 2)$.
Then there exist neighborhoods $U(4)$, $V(2)$ *and unique* $f\colon U \to V$ *such that*

- f *is continuously differentiable and* $f(4) = 2$.
- $\Phi(x, f(x)) = 0$ *for all* $x \in U$.
- $f'(x) = -\frac{\partial_x \Phi(x, f(x))}{\partial_y \Phi(x, f(x))} = \frac{1}{2f(x)}$

Implicit Function Theorem

This can be generalized to the

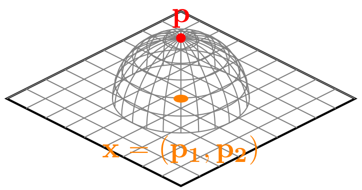
Theorem 3 (Implicit Function Theorem). *Let $\Phi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a cont. differentiable mapping which satisfies $\Phi(x_0, y_0) = 0$ for a $(x_0, y_0) \in \mathbb{R}^{m+n}$ and $\partial_y \Phi(x_0, y_0)$ is invertible.*
Then there exist neighborhoods $U(x_0)$, $V(y_0)$ and a continuously differentiable function $f\colon U \to V$ such that

$$\Phi(x, f(x)) = 0 \qquad \forall x \in U$$

and

$$\begin{aligned} f(x_0) &= y_0 \\ f'(x) &= -\left(\partial_y \Phi(x, f(x))\right)^{-1} \partial_x \Phi(x, f(x)) \qquad \forall x \in U \end{aligned}$$

Implicit Function Theorem (Example)



Note that we can also use the implicit function theorem if we are not looking for the inverse of a function.

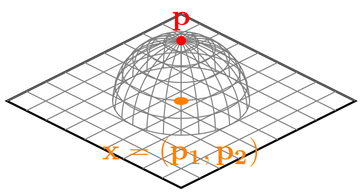
The points $p = (p_1, p_2, p_3)$ on the unit sphere satisfy

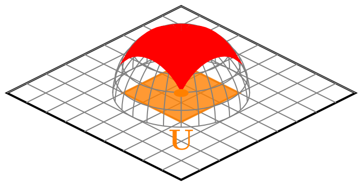
$$p_1^2 + p_2^2 + p_3^2 = 1$$

If we use the notation $x = (p_1, p_2)$ and $y = p_3$, the requirements for the implicit function theorem are satisfied for $x_0 = (0, 0)$, $y_0 = 1$ as well as

$$\Phi(x, y) = \|x\|^2 + y^2 - 1$$

Implicit Function Theorem (Example)





In particular, there exists a neighborhood $U(x_0) \subset \mathbb{R}^2$ and a mapping $f\colon U \to \mathbb{R}$ such that

$$\begin{aligned} \varphi\colon U &\to \mathbb{R}^3 \\ x &\mapsto (x, f(x)) \end{aligned}$$

maps the 2D region U onto a part of the sphere.

Functions like φ can be used to map a subset of a two-dimensionale linear space onto a subset of a two-dimensional curved space. These curved spaces are called **manifolds**.