Analysis of 3D Shapes (IN2238)

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Summer Semester 2017

3. Manifolds and Shapes

Dimension

Dimension of Linear Spaces

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Even for (linear) $\mathbb{R}$-vector spaces, the definition can be quite involved:

$$\dim(U) = \min_{\substack{B \subset U \\ \operatorname{span}(B)=U}} \#B$$

$$\operatorname{span}(B) = \left\{ x \,\middle|\, x = \sum_{b \in B} \lambda_b \cdot b, \lambda \in \mathbb{R}^B \right\}$$

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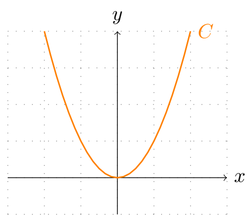
For linear vector spaces U, it suffices to find a linear bijection $\Phi : \mathbb{R}^d \to U$ in order to prove that U is a vector space of dimension d.

Here, U can be **an arbitrary $\mathbb{R}$-vector space**. It does not need to be **represented as a subset of an $\mathbb{R}^N$**.

Dimension of Curves



Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 | y = x^2\}$$

is not a linear space, we like to think of it as a one-dimensional object. Why?

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For what kind of sets C can we define a "curved" dimension?

What kind of functions φ guarantee a unique dimension?

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Continuous Mappings

Dimension Manifold Submanifold Tangent Space

Definition 1 (Open Set)**.** A set $O \subset \mathbb{R}^n$ is called open iff

$$x \in O \qquad \Rightarrow \qquad \exists \varepsilon > 0 : B_\varepsilon(x) \subset O,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n | \, \|x - y\| < \varepsilon\}$ is a ball of radius ε centered at $x \in \mathbb{R}^n$.

Definition 2 (Relatively Open Set)**.** Given a subset $X \subset \mathbb{R}^n$, we call $O \subset X$ relatively open iff there exists an open set $\hat{O} \subset \mathbb{R}^n$ of $\mathbb{R}^n$ such that

$$O = \hat{O} \cap X$$

The set $\mathcal{T}(X)$ of all relatively open subsets is called the topology of X.

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Definition 3 (Continuous Mappings)**.** Given subsets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, we call a mapping $f\colon X \to Y$ continuous iff

$$O \in \mathcal{T}(Y) \qquad \Rightarrow \qquad f^{-1}(O) \in \mathcal{T}(X)$$

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Homeomorphism

Dimension Manifold Submanifold Tangent Space

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Theorem 3 (Brouwer, 1911). *The existence of a homeomorphism $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ implies $m = n$.*

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Definition 5. A bijection $\varphi\colon X \to Y$ is called a **C^k-diffeomorphism** iff φ and φ^{-1} are C^k-functions, *i.e.*, for all $i \leqslant k$ exist the i-th derivatives of these functions and they are continuous. C^∞-diffeomorphisms are called **(smooth) diffeomorphisms**.

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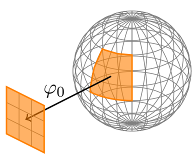
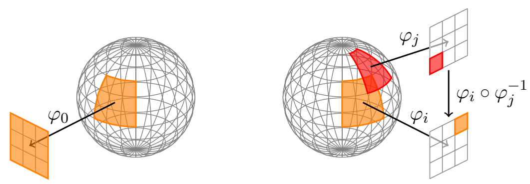
Solution: Concept of **manifolds** and **sub-manifolds**.

Manifold



Definition 6. (M_0, φ_0) is called an **n-dimensional chart** of M iff $\varphi_0\colon M_0 \to U_0$ is a homeomorphism between the open sets $M_0 \subset M$ and $U_0 \subset \mathbb{R}^n$.



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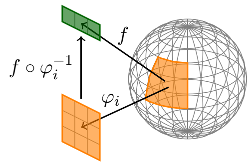
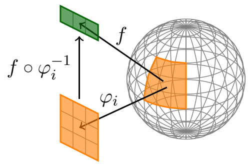
Definition 7. A collection $(M_i, \varphi_i)_{i \in \mathcal{I}}$ of charts is called a **C^k atlas** iff $M = \bigcup_{i \in \mathcal{I}} M_i$ and for any two charts φ_i and φ_j, the mapping $\varphi_i \circ \varphi_j^{-1}$ is a C^k-diffeomorphism between $\varphi_j(M_i \cap M_j)$ and $\varphi_i(M_i \cap M_j)$.

Definition 8. A set M with a C^∞ atlas $(M_i, \varphi_i)_{i \in \mathcal{I}}$ is called a **(smooth) manifold**.



Definition 8. A set M with a C^∞ atlas $(M_i, \varphi_i)_{i \in \mathcal{I}}$ is called a **(smooth) manifold**.

Definition 9. Given a manifold M, a function $f\colon M \to \mathbb{R}^d$ is called **smooth** iff for all charts (M_i, φ_i), the function $f \circ \varphi_i^{-1}\colon \varphi_i(M_i) \to \mathbb{R}^d$ is smooth.



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Whether a function is smooth depends very much on the chosen atlas. Therefore, we are more interested in submanifolds.

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Let $(M_i)_{i \in \mathcal{I}}$ be a collection of open sets $M_i \subset \mathbb{R}^n$ that cover $\mathbb{R}^n$. Choices are:

$$\mathcal{I} = \mathbb{Z}^n \qquad M_{(i_1,\dots,i_n)} = \left\{x \in \mathbb{R}^n \,\middle|\, \|x - i\| < \sqrt{n}\right\}$$
$$\mathcal{I} = \{0\} \qquad M_0 = \mathbb{R}^n$$

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Given these sets, the charts can be choosen as (M_i, id_{M_i}) and for overlapping charts, we obtain the diffeomorphism

$$\varphi_i \circ \varphi_j^{-1}\colon M_i \cap M_j \to M_i \cap M_j$$
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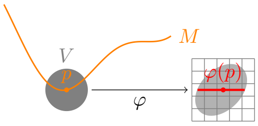
With these atlases, we obtain that the "smooth functions" f on the "manifold" $\mathbb{R}^n$ are exactly the functions $C^\infty(\mathbb{R}^n)$.

Submanifold

Smooth Submanifold



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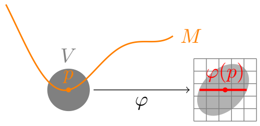

Definition 10. A subset $M \subset \mathbb{R}^n$ is a **(smooth) submanifold** of dimension m iff for every point $p \in M$, there exists a chart (V, φ) of $\mathbb{R}^n$ such that

- $p \in V$.
- $\varphi(M \cap V) = \mathbb{R}^m \cap \varphi(V)$.

We call $n - m$ the **co-dimension** of M.

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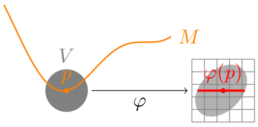

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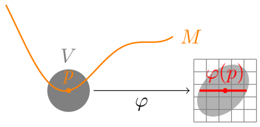

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Lemma 1. *A subset $M \subset \mathbb{R}^n$ together with smooth* ***coordinate mappings*** *$(x_i, U_i)_{i \in \mathcal{I}}$ is a smooth submanifold of dimension m if the following holds:*

- *All U_i are open subsets of $\mathbb{R}^m$.*
- *$M = \bigcup_{i \in \mathcal{I}} x_i(U_i)$.*
- *For all $u \in U_i$, x_i is smooth and $Dx_i(u) \in \mathbb{R}^{n \times m}$ is of* ***maximal rank*** *m.*

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Proof. Given $p \in M$, we choose $i \in \mathcal{I}$ and $\hat{u} \in \mathbb{R}^m$ such that $p = x_i(\hat{u})$. Since $Dx_i(\hat{u})$ is of maximal rank, we can find a matrix $A_0 \in \mathbb{R}^{n \times (n-m)}$ such that $A := \begin{pmatrix} Dx_i(\hat{u}) & A_0 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is of maximal rank n, *i.e.*, invertible.

Proof (Cont.) As a result, we can define the smooth function

$$\psi : U_i \times \mathbb{R}^{n-m} \to \mathbb{R}^n$$
$$(u_1,\dots,u_m,v_1,\dots,v_{n-m}) \mapsto x_i(u) + A_0 \cdot v$$

with $D\psi(\hat{u},0) = \begin{pmatrix} Dx_i(\hat{u}) & A_0 \end{pmatrix} = A$. Using the *Inverse Function Theorem* proves the lemma for $\varphi := \psi^{-1}$. □

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Now we can prove that

$$C := \{(x,y) \in \mathbb{R}^2 | y = x^2\}$$

is a manifold of dimension 1. How?

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In practice, it is often difficult to define different charts or coordinate functions. Instead, we like to define the manifold M by formulating certain constraints, *e.g.*,

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Lemma 2. *Given a function* $f\colon \mathbb{R}^n \to \mathbb{R}^k$ *and a* **regular value** $c \in \mathbb{R}^k$, i.e.,

$$x \in f^{-1}(c) \quad \Rightarrow \quad \operatorname{rank}(Df(x)) = k.$$

Then, $M := f^{-1}(c)$ *is a submanifold of co-dimension* k.

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Example. $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f(x) = \|x\|^2$ and $c > 0$.

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Proof. Let $p \in M \subset \mathbb{R}^n$. Since $Df(p)$ is of rank k, we can find k columns of $Df(p)$ that are linear independent. W.l.o.g. we assume that these k columns are the last k. Thus, the function $f\colon \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies the *Implicit Function Theorem* with respect to $(x_0, y_0) = p$.

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The implicit function $\varphi\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ defines a coordinate mapping $x\colon \mathbb{R}^k \to \mathbb{R}^n$ via $x(u) := (u, \varphi(u))$, which proves the lemma. □

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Given a point $p \in M$ the created coordinate mapping x satisfies

$$Dx(p) = \begin{pmatrix} \mathrm{id} \\ -\left[\frac{\partial f}{\partial y}(p)\right]^{-1} \frac{\partial f}{\partial x}(p) \end{pmatrix}$$

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For now we will focus on an equivalence relation that uses a combination of rotation, translation and scaling to define equivalent objects.

Tangent Space

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Definition 13 (Tangent Space). Let $M \subset \mathbb{R}^n$ be a submanifold of dimension $m \leqslant n$ that is given via coordinate functions $(x_i, U_i)_{i\in\mathcal{I}}$. Given $i \in \mathcal{I}$ such that $p = x_i(u)$, we define the **tangent space** T_pM of M at the position p as

$$T_pM := \{Dx_i(u)\cdot v | v \in \mathbb{R}^m\} \qquad \left[= \mathrm{Im}(Dx_i(u))\right]$$

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Proof. Excercise. □

Lemma 4 (Tangent Space). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function with regular value $c \in \mathbb{R}^k$ and $M := f^{-1}(c)$ the manifold with respect to this value. For every $p \in M$ we have*

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Choosing $y_2 = [\frac{\partial f}{\partial y}(p)]^\top \cdot v$ leads to $y_1 = [\frac{\partial f}{\partial x}(p)]^\top \cdot v$, which proves $T_pM = \operatorname{Im}(Df(p)^\top)^\perp = \ker(Df(p))$. □

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