

Analysis of 3D Shapes (IN2238)

Frank R. Schmidt
Matthias Vestner

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3. Manifolds and Shapes

Dimension

Dimension of Linear Spaces

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$$\text{span}(B) = \left\{ x \mid x = \sum_{b \in B} \lambda_b \cdot b, \lambda \in \mathbb{R}^B \right\}$$

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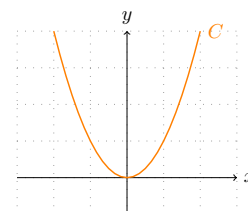
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For linear vector spaces U , it suffices to find a linear bijection $\Phi : \mathbb{R}^d \rightarrow U$ in order to prove that U is a vector space of dimension d .

Here, U can be an **arbitrary \mathbb{R} -vector space**. It does not need to be **represented as a subset of an \mathbb{R}^N** .

Dimension of Curves



Even though the curve

$$C := \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$$

is not a linear space, we like to think of it as a one-dimensional object. Why?



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For what kind of sets C can we define a "curved" dimension?

What kind of functions φ guarantee a unique dimension?



Definition 1 (Open Set). A set $O \subset \mathbb{R}^n$ is called open iff

$$x \in O \quad \Rightarrow \quad \exists \varepsilon > 0 : B_\varepsilon(x) \subset O,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}$ is a ball of radius ε centered at $x \in \mathbb{R}^n$.

Definition 2 (Relatively Open Set). Given a subset $X \subset \mathbb{R}^n$, we call $O \subset X$ relatively open iff there exists an open set $\hat{O} \subset \mathbb{R}^n$ of \mathbb{R}^n such that

$$O = \hat{O} \cap X$$

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Definition 3 (Continuous Mappings). Given subsets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, we call a mapping $f: X \rightarrow Y$ continuous iff

$$O \in \mathcal{T}(Y) \quad \Rightarrow \quad f^{-1}(O) \in \mathcal{T}(X)$$



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Definition 4. A bijection $\varphi: X \rightarrow Y$ is called a **homeomorphism** iff φ and φ^{-1} are continuous.

Homeomorphism

Dimension
Manifold
Submanifold
Tangent Space

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Theorem 3 (Brouwer, 1911). *The existence of a homeomorphism $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ implies $m = n$.*

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Diffeomorphism

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In practice, it is often difficult to check whether a function is continuous. To check differentiability is on the other hand easier. Thus, we would like to extend the idea of homeomorphisms to the class of differentiable functions.

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Definition 5. A bijection $\varphi : X \rightarrow Y$ is called a **C^k -diffeomorphism** iff φ and φ^{-1} are C^k -functions, i.e., for all $i \leq k$ exist the i -th derivatives of these functions and they are continuous. C^∞ -diffeomorphisms are called **(smooth) diffeomorphisms**.

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Solution: Concept of **manifolds** and **sub-manifolds**.

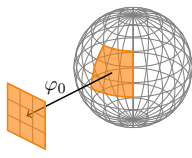
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Manifold

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Chart and Atlas

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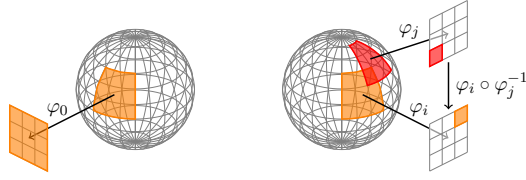


Definition 6. (M_0, φ_0) is called an **n -dimensional chart** of M iff $\varphi_0 : M_0 \rightarrow U_0$ is a homeomorphism between the open sets $M_0 \subset M$ and $U_0 \subset \mathbb{R}^n$.

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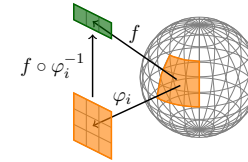
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Definition 7. A collection $(M_i, \varphi_i)_{i \in I}$ of charts is called a **C^k atlas** iff $M = \bigcup_{i \in I} M_i$ and for any two charts φ_i and φ_j , the mapping $\varphi_i \circ \varphi_j^{-1}$ is a C^k -diffeomorphism between $\varphi_j(M_i \cap M_j)$ and $\varphi_i(M_i \cap M_j)$.

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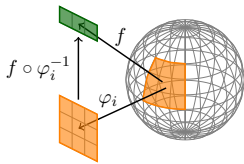


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Whether a function is smooth depends very much on the chosen atlas. Therefore, we are more interested in submanifolds.



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Let $(M_i)_{i \in \mathcal{I}}$ be a collection of open sets $M_i \subset \mathbb{R}^n$ that cover \mathbb{R}^n . Choices are:

$$\begin{aligned} \mathcal{I} &= \mathbb{Z}^n & M_{(i_1, \dots, i_n)} &= \{x \in \mathbb{R}^n \mid \|x - i\| < \sqrt{n}\} \\ \mathcal{I} &= \{0\} & M_0 &= \mathbb{R}^n \end{aligned}$$



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Given these sets, the charts can be chosen as (M_i, id_{M_i}) and for overlapping charts, we obtain the diffeomorphism

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}: M_i \cap M_j &\rightarrow M_i \cap M_j \\ p &\mapsto p \end{aligned}$$



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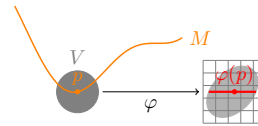
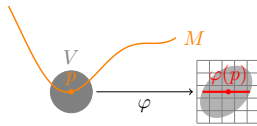
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With these atlases, we obtain that the "smooth functions" f on the "manifold" \mathbb{R}^n are exactly the functions $C^\infty(\mathbb{R}^n)$.

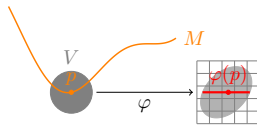




Definition 10. A subset $M \subset \mathbb{R}^n$ is a **(smooth) submanifold** of dimension m iff for every point $p \in M$, there exists a chart (V, φ) of \mathbb{R}^n such that

- $p \in V$.
- $\varphi(M \cap V) = \mathbb{R}^m \cap \varphi(V)$.

We call $n - m$ the **co-dimension** of M .

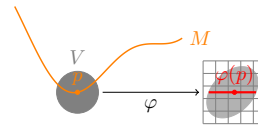


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- All U_i are open subsets of \mathbb{R}^m .
- $M = \bigcup_{i \in \mathcal{I}} x_i(U_i)$.
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Proof. Given $p \in M$, we choose $i \in \mathcal{I}$ and $\hat{u} \in \mathbb{R}^m$ such that $p = x_i(\hat{u})$. Since $Dx_i(\hat{u})$ is of maximal rank, we can find a matrix $A_0 \in \mathbb{R}^{n \times (n-m)}$ such that $A := (Dx_i(\hat{u}) \ A_0) \in \mathbb{R}^{n \times n}$ is of maximal rank n , i.e., invertible.



Proof (Cont.) As a result, we can define the smooth function

$$\psi : U_i \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$$

$$(u_1, \dots, u_m, v_1, \dots, v_{n-m}) \mapsto x_i(u) + A_0 \cdot v$$

with $D\psi(\hat{u}, 0) = (Dx_i(\hat{u}) \ A_0) = A$. Using the *Inverse Function Theorem* proves the lemma for $\varphi := \psi^{-1}$. \square



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Now we can prove that

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is a manifold of dimension 1. How?



In practice, it is often difficult to define different charts or coordinate functions. Instead, we like to define the manifold M by formulating certain constraints, e.g.,

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Lemma 2. Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and a **regular value** $c \in \mathbb{R}^k$, i.e.,

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Example. $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $f(x) = \|x\|^2$ and $c > 0$.



Proof. Let $p \in M \subset \mathbb{R}^n$. Since $Df(p)$ is of rank k , we can find k columns of $Df(p)$ that are linear independent. W.l.o.g. we assume that these k columns are the last k . Thus, the function $f : \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ satisfies the *Implicit Function Theorem* with respect to $(x_0, y_0) = p$.



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Given a point $p \in M$ the created coordinate mapping x satisfies

$$Dx(p) = \begin{pmatrix} \text{id} \\ -\left[\frac{\partial f}{\partial y}(p)\right]^{-1} \frac{\partial f}{\partial x}(p) \end{pmatrix}$$



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Definition 11 (Object). An **object** of dimension d is an open subset $X \subset \mathbb{R}^d$ such that its boundary $B := \partial X$ is a submanifold of dimension $d - 1$. We will use the notation \mathcal{O}^d for the space of all objects of dimension d .



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Definition 12 (Shape). Given an equivalence relation \sim of \mathcal{O}^d , the equivalence class $[O]$ of an object $O \in \mathcal{O}^d$ is its shape. We call the set of all shapes the **shape space** \mathcal{O}^d / \sim .



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Some authors only study the boundary of an object.

Definition 12 (Shape). Given an equivalence relation \sim of \mathcal{O}^d , the equivalence class $[O]$ of an object $O \in \mathcal{O}^d$ is its shape. We call the set of all shapes the **shape space** \mathcal{O}^d / \sim .

For now we will focus on an equivalence relation that uses a combination of rotation, translation and scaling to define equivalent objects.

Tangent Space

Since we want to focus on smooth submanifolds M of dimension m , we like to approximate the direct vicinity of a point $p \in M$ with a linear vector space of dimension m . This leads to the concept of the tangent space.

Tangent Space

Dimension Manifold Submanifold **Tangent Space**

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Definition 13 (Tangent Space). Let $M \subset \mathbb{R}^n$ be a submanifold of dimension $m \leq n$ that is given via coordinate functions $(x_i, U_i)_{i \in \mathcal{I}}$. Given $i \in \mathcal{I}$ such that $p = x_i(u)$, we define the **tangent space** $T_p M$ of M at the position p as

$$T_p M := \{Dx_i(u) \cdot v \mid v \in \mathbb{R}^m\} \quad [= \text{Im}(Dx_i(u))]$$

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Lemma 3 (Tangent Space). *The tangent space $T_p M \subset \mathbb{R}^n$ is a linear subspace and does not depend on the choice of the coordinate map (x_i, U_i) .*

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Proof. Exercise. □

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Lemma 4 (Tangent Space). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a smooth function with regular value $c \in \mathbb{R}^k$ and $M := f^{-1}(c)$ the manifold with respect to this value. For every $p \in M$ we have*

$$T_p M := \{v \in \mathbb{R}^n \mid Df(p) \cdot v = 0\} \quad [= \ker(Df(p))]$$

Proof. The coordinate mapping x that we constructed for an implicitly defined manifold is $Dx(u) = \begin{pmatrix} \text{id} \\ -[\frac{\partial f}{\partial y}(p)]^{-1} \frac{\partial f}{\partial x}(p) \end{pmatrix}$.

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$$\Rightarrow 0 = y_1 - \left[\frac{\partial f}{\partial x}(p) \right]^\top \left[\frac{\partial f}{\partial y}(p) \right]^{-\top} y_2$$

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Choosing $y_2 = \left[\frac{\partial f}{\partial y}(p) \right]^\top \cdot v$ leads to $y_1 = \left[\frac{\partial f}{\partial x}(p) \right]^\top \cdot v$

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$$\begin{aligned} y \in T_p M^\perp &= \text{Im}(Dx(u))^\perp = \ker(Dx(u)^\top) \\ \Rightarrow 0 &= y_1 - \left[\frac{\partial f}{\partial x}(p) \right]^\top \left[\frac{\partial f}{\partial y}(p) \right]^{-\top} y_2 \end{aligned}$$

Choosing $y_2 = \left[\frac{\partial f}{\partial y}(p) \right]^\top \cdot v$ leads to $y_1 = \left[\frac{\partial f}{\partial x}(p) \right]^\top \cdot v$, which proves $T_p M = \text{Im}(Df(p)^\top)^\perp = \ker(Df(p))$. \square



Dimension

- Peano, *Sur une courbe, qui remplit toute une aire plane*, 1890, Math. Annalen (36), 157–160.



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Smooth Manifolds



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Smooth Manifolds

- Poincaré, *Analysis Situs*, 1895, Journal de l'École Polytechnique.