Analysis of 3D Shapes (IN2238)

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9. First Fundamental Form

Coordinate Transform

Linear Mapping

Given the $\mathbb{R}$-vector spaces U and V, a mapping $L\colon U \to V$ is a **linear mapping** if the following holds:

$$L(u+v) = L(u) + L(v) \qquad \forall u, v \in U$$
$$L(\lambda u) = \lambda L(u) \qquad \forall \lambda \in \mathbb{R}, u \in U$$

Is X a basis of the n-dimensional vector space U and Y a basis of the m-dimensional vector space V, we obtain

$$L(x_j) = \sum_{i=1}^{m} a_{ij} y_i$$

$A \in \mathbb{R}^{m\times n}$ is then called the **representing matrix** of L with respect to the bases X and Y and we write:

$$\mathcal{M}_Y^X(L) = A.$$

Change of Linear Bases

If we have $V = \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, the matrix-vector multiplication defines a linear mapping:

$$L\colon \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

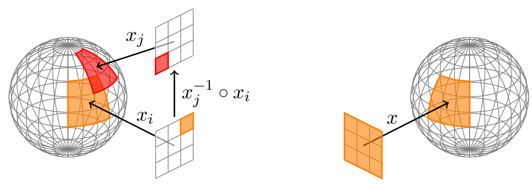
Let us assume that we want to **change the bases** of $\mathbb{R}^n$. To that end, both X and Y can be written in matrix form and we have

$$\mathcal{M}_Y^X(L) = Y^{-1} \cdot A \cdot X$$

Thus, there is a subtle difference between linear mappings L and matrices A. A is a representation of L that also takes the specific bases into account.

We say that two matrices A and B are **similar**, if there exists an invertible matrix X such that $B = X^{-1} \cdot A \cdot X$.

Coordinate Changes



We think of a coordinate mapping as a C^1 function

$$x_i\colon U_i \to M \subset \mathbb{R}^{d+1} \quad \text{with} \quad x_i(U_i) \subset M \quad U_i \subset \mathbb{R}^d$$

where M describes a manifold of dimension d and the diffeomorphism $x_j^{-1} \circ x_i$ tells us how one coordinate system can be transformed into another.

Goal: We like to measure some quantities of M directly on the domain U_i. These quantities should be independent of a coordinate mapping x_i.

Angles, Lengths and Areas

Important geometric quantities that we like to measure are

Angles In practice, we are not so much interested in the angle α itself, but rather in $\cos(\alpha)$. In particular, we would like to determine whether two lines that pass through a point p are orthogonal to one another.
In order to measure $\cos(\alpha)$ we need something like a scalar product.

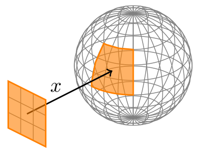
Length We already saw that the length of a curve can be computed by integration. We therefore need to translate the line integral on $M \subset \mathbb{R}^{d+1}$ into a line integral on the domain $U \subset \mathbb{R}^d$.

Area Usually integration is considered in the context of computing the size of certain areas. It is therefore natural to also transform a surface integral on M into a surface integral on U.

These problems can be addressed with the so called **First Fundamental Form**. It is also called the **Riemannian Metric** or the **metrical tensor**.

First Fundamental Form

First Fundamental Form



Given the coordinate mapping $x\colon U \to M \subset \mathbb{R}^{d+1}$, the **first fundamental form** is defined as

$$g\colon U \to \mathbb{R}^{d\times d} \qquad u \mapsto Dx(u)^\top Dx(u)$$

The matrix $g(u)$ is symmetric and positive-definite, *i.e.*,

$$g(u)^\top = g(u) \qquad \langle X, g(u)\cdot X\rangle > 0 \qquad \forall X \neq 0$$

Generalized Scalar Product

Given the vector space V, we call $b\colon V\times V \to \mathbb{R}$ a scalar product if:

$$\begin{aligned} b(\cdot, X), b(X,\cdot) & \quad \text{are linear for all } X \in V \\ b(X,Y) = b(Y,X) & \quad \text{for all } X, Y \in V \\ b(X,X) > 0 & \quad \text{for all } X \neq 0 \end{aligned}$$

Scalar products on $\mathbb{R}^n$ are represented by a symmetric, pos.-definite $A \in \mathbb{R}^{n\times n}$

$$b(X,Y) := \langle X, A\cdot Y\rangle$$

and are denoted as $\langle X, Y\rangle_A$.

Therefore, the Riemannian metric can be seen as a continuously changing scalar product on the domain U and is sometimes denoted as $\langle X, Y\rangle_{g(u)}$.

Scalar Product on the Tangent Space

Let us assume the coordinate map $x\colon U \to M$ and the first fundamental form $g\colon U \to \mathbb{R}^{d\times d}$.
Further, let us assume that two curves $\gamma_{1,2}\colon (-\varepsilon, \varepsilon) \to U$ are given in the parameter domain U that pass through the same point $\gamma_1(0) = u = \gamma_2(0)$.
Now let $X := \gamma_1'(0) \in \mathbb{R}^d$ and $Y := \gamma_2'(0) \in \mathbb{R}^d$.

The curves γ_i define curves $c_i := x \circ \gamma_i$ on the manifold M and pass through the point $p = x(u) \in M$.

The curves define tangent vectors in T_pM via

$$v_i := c_i'(0) = Dx(u)\cdot \gamma_i'(0) \in T_pM \subset \mathbb{R}^{d+1}$$

Thus, we can move the computation on the tangent space T_pM back onto U

$$\langle v_1, v_2\rangle = \langle Dx(u)X, Dx(u)Y\rangle = \langle X, Y\rangle_{g(u)}$$

Measuring Angles

In summary, we can measure the cosine of the angle α between $c_1 = x\circ\gamma_1$ and $c_2 = x \circ \gamma_2$ in means of $X = \gamma_1'(0)$ and $Y = \gamma_2'(0)$:

$$\cos(\alpha) = \frac{\langle v_1, v_2\rangle}{\|v_1\|\,\|v_2\|} = \frac{\langle X, Y\rangle_{g(u)}}{\|X\|_{g(u)}\,\|Y\|_{g(u)}}$$

Thus, we are able to measure $\cos(\alpha)$ by just looking at g. We call every quantity that can be measured in that way as **intrinsic**. They do not depend on the surrounding space, but only on measurements "inside" of the manifold.

A coordinate mapping $x\colon U \to M$ is called **conformal** if every angle measurement in U coincides with the angle measurement on M.

The Riemannian metric g of a conformal coordinate mapping $x\colon U \to M$ is the identity matrix multiplied with a scalar $r\colon U \to \mathbb{R}$ that depends on the location $u \in U$ of the parametrization domain.

Example (Cylindrical Coordinates)

A cylinder $Z = \mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3$ can be parametrized via the following mapping

$$x\colon (0,\pi)\times\mathbb{R} \to Z \qquad (\alpha, r) \mapsto \begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\\ r\end{pmatrix}$$

Therefore we have

$$Dx(\alpha, r) = \begin{pmatrix} -\sin(\alpha) & 0\\ \cos(\alpha) & 0 \\ 0 & 1\end{pmatrix} \qquad g(\alpha, r) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

This means that there is no observable, intrinsic difference between a patch on a cylinder and a patch of the 2D space if we use this specific parametrization.

Example (Polar Coordinates)

We can also reparametrize $\mathbb{R}^2$ itself.
Common coordinates are the **polar coordinates**

$$x\colon (0, 2\pi)\times\mathbb{R}_+ \to \mathbb{R}^2 \qquad (\alpha, r) \mapsto \begin{pmatrix} r\cos(\alpha)\\ r\sin(\alpha)\end{pmatrix}$$

Therefore we have

$$Dx(\alpha, r) = \begin{pmatrix} -r\sin(\alpha) & \cos(\alpha)\\ r\cos(\alpha) & \sin(\alpha)\end{pmatrix} \qquad g(\alpha, r) = \begin{pmatrix} r^2 & 0\\ 0 & 1\end{pmatrix}$$

For these coordinates the angle between e_1 and e_2 is always 90°, but the mapping is not conformal. Why?

Example (Spherical Parametrization)

It is common to use the following parametrization for the sphere $\mathbb{S}^2$:

$$x\colon (0,2\pi)\times(0,\pi) \to \mathbb{S}^2 \qquad (\alpha, \beta) \mapsto \begin{pmatrix}\cos(\alpha)\sin(\beta)\\ \sin(\alpha)\sin(\beta)\\ \cos(\beta)\end{pmatrix}$$

Therefore we have

$$Dx(\alpha, \beta) = \begin{pmatrix} -\sin(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta)\\ \cos(\alpha)\sin(\beta) & \sin(\alpha)\cos(\beta)\\ 0 & -\sin(\beta)\end{pmatrix} \qquad g(\alpha, r) = \begin{pmatrix}\sin(\beta)^2 & 0\\ 0 & 1\end{pmatrix}$$

Length Computation

In the following we want to revise the length computation of a curve.

To this end, let $x\colon U \to M$ be a coordinate map, $\gamma\colon [0;1] \to U$ a curve in the parametrization domain U and

$$c\colon [0;1] \to M \qquad c(t) := x \circ \gamma(t)$$

the curve on the manifold M whose length we like to measure.

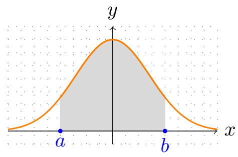
The length of c can be computed via

$$\text{length}(c) = \int_0^1 \|\dot c(t)\|\,\mathrm{d}t = \int_0^1 \|Dx(\gamma(t))\cdot\dot\gamma(t)\|\,\mathrm{d}t = \int_0^1 \|\dot\gamma(t)\|_{g(\gamma(t))}\,\mathrm{d}t$$

We can express $\text{length}(c)$ with γ as long as we take the Riemannian metric g into account. Thus, $\text{length}(c)$ is an intrisic quantity of M.

Integration

Integral (Riemann)



If a function $f\colon [a,b] \to \mathbb{R}$ can be integrated, we obtain

$$\int_a^b f(s)\,\mathrm{d}s = \lim_{N\to\infty} \sum_{i=1}^{N} f\left(a + \frac{b-a}{N}\right) \cdot \frac{b-a}{N}$$

This means in particular that there is a difference between $\int_a^b$ and $\int_b^a$. This integral is called the **Riemann integral**.

Integral (Lebesgue)

Lebesgue proposed a different integral that does not take the order of an interval's boundary into account

$$\int_{[a,b]} f(s)\,\mathrm{d}s = \lim_{N\to\infty} \sum_{i=1}^{N} f\left(a + \frac{b-a}{N}\right) \cdot \left|\frac{b-a}{N}\right|$$

The main idea of the **Lebesgue integral** is to **measure the size of sets** and combine this with the integral notation of Riemann.

In general, every function f that can be integrated with respect to Riemann can also be integrated with respect to Lebesgue.

The mathematical contribution of Lebesgue was that there are sets that can be measured even though they cannot be represented by a finite union of intervals. Therefore, the Lebesgue integral is a generalization of the Riemann integral.

Substitution Rule

The **substitution rule** for integration is a consequence of the **chain rule** and the **main theorem of calculus**:

If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function, we obtain:

$$\int_{\varphi(a)}^{\varphi(b)} f(s)\,\mathrm{d}s = \int_a^b f \circ \varphi(s) \cdot \varphi'(s)\,\mathrm{d}s$$

for the integral with respect to Riemann.

For the integral with respect to Lebesgue we obtain instead

$$\int_{\varphi(S)} f(s)\,\mathrm{d}s = \int_S f \circ \varphi(s) \cdot |\varphi'(s)|\,\mathrm{d}s$$

with respect to any set S that can be measured.

Substitution Rule in Higher Dimension

If $\varphi\colon \mathbb{R}^k \to \mathbb{R}^k$ is a C^1 function, we obtain the following substitution rule

$$\int_{\varphi(S)} f(s)\,\mathrm{d}s = \int_S f \circ \varphi(s) \cdot |D\varphi(s)|\,\mathrm{d}s$$

with respect to any set S that can be measured. What is $|D\varphi(s)|$?

$D\varphi(s) \in \mathbb{R}^{k\times k}$ describes a linear transformation of the unit vectors $e_1, \dots, e_k$.

Therefore, we have:

$$\begin{aligned} |D\varphi(s)| &:= \mathrm{vol}(D\varphi(s)e_1, \dots, D\varphi(s)e_k) \\ &= |\det(D\varphi(s))| \end{aligned}$$

If we work with manifolds, we also would like to compute $|A|$ for any matrix $A \in \mathbb{R}^{k\times m}$ with $m \leqslant k$.

Integral (Example)

Let us consider the integration problem

$$I = \int_{\mathbb{R}} \exp\left(-s^2\right)\mathrm{d}s\,.$$

Then we have

$$\begin{aligned} I^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(-s_1^2\right)\mathrm{d}s_1 \exp\left(-s_2^2\right)\mathrm{d}s_2 = \int_{\mathbb{R}^2} \exp\left(-(s_1^2 + s_2^2)\right)\mathrm{d}s \\ &= \int_0^{2\pi}\int_0^{\infty} \exp(-r^2) \cdot |r|\,\mathrm{d}r\mathrm{d}\varphi = \pi \end{aligned}$$

This means we have

$$I = \int_{\mathbb{R}} \exp\left(-s^2\right)\mathrm{d}s = \sqrt{\pi}$$

Volume of a Matrix

For any matrix $A \in \mathbb{R}^{k\times k}$ we can compute its volume as

$$|A| = |\det(A)|$$

Let us now assume that $A \in \mathbb{R}^{k\times m}$ is given. We can find an orthonormal extension of A, *i.e.*,

$$\hat{A} = (A\ Q) \in \mathbb{R}^{n\times n} \qquad Q \in \mathbb{R}^{k\times(k-m)} \qquad Q^\top Q = I_{k-m}$$

Since Q only multiplies a volume of size 1 orthogonal to A, we can define

$$\begin{aligned} |A| &:= \left|\hat{A}\right| = \det\left(\hat{A}^\top \cdot \hat{A}\right)^{\frac{1}{2}} \\ &= \det\begin{pmatrix} A^\top A & 0 \\ 0 & I_{k-m} \end{pmatrix}^{\frac{1}{2}} = \sqrt{\det(A^\top A)} \end{aligned}$$

Integrating on a Manifold

If $\varphi\colon \mathbb{R}^k \to \mathbb{R}^m$ is a C^1 function, we obtain the following substitution rule

$$\int_{\varphi(S)} f(s)\,\mathrm{d}s = \int_S f \circ \varphi(s) \cdot \det\left(D\varphi(s)^\top D\varphi(s)\right)^{\frac{1}{2}}\mathrm{d}s$$

with respect to any set S that can be measured.

Let us now assume that $x_i\colon U_i \to M$ is a coordinate function and $f\colon M \to \mathbb{R}$ is a scalar function on M. Then we obtain for $M_i = x(U_i)$

$$\begin{aligned} \int_{M_i} f(p)\,\mathrm{d}p &= \int_{U_i} f \circ x_i(u) \cdot \det\left(Dx_i(u)^\top Dx_i(u)\right)^{\frac{1}{2}}\mathrm{d}u \\ &= \int_{U_i} f \circ x_i(u) \cdot \sqrt{\det g(u)}\,\mathrm{d}u \end{aligned}$$

Therefore, integration of a scalar valued function can be done intrinsically.

Integral (Example)

We like to compute the area of a sphere of radius R by using the standard parametrization ($\alpha \in (0, 2\pi)$, $\beta \in (0, \pi)$):

$$x(\alpha, \beta) = \begin{pmatrix} R\cos(\alpha)\sin(\beta) \\ R\sin(\alpha)\sin(\beta) \\ R\cos(\beta) \end{pmatrix} \qquad g(\alpha, \beta) = \begin{pmatrix} R^2\sin(\beta)^2 & 0 \\ 0 & R^2 \end{pmatrix}$$

Then we have

$$\begin{aligned} \text{area}(R \cdot \mathbb{S}^2) &= \int_{R \cdot \mathbb{S}^2} 1 \,\mathrm{ds} = \int_0^{2\pi} \int_0^{\pi} R^2 \sin(\beta) \mathrm{d}\beta \mathrm{d}\alpha \\ &= \int_0^{2\pi} R^2 \left[\cos(0) - \cos(\pi)\right] \mathrm{d}\alpha = 4\pi R^2 \end{aligned}$$