Analysis of 3D Shapes (IN2238)

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13. Gauss Curvature

Theorema Egregium

Summary and Notations

Given a coordinate map $x\colon U \to M$ and the Gauss map $G\colon M \to \mathbb{S}^2$ of the surface $M \subset \mathbb{R}^3$, we have for $p = x(u)$ and $v_1, v_2 \in T_pM$

- $\{\partial_1 x(u), \partial_2 x(u)\}$ is a base of the tangent plane T_pM.
- $g_{ij}(u) = \langle \partial_i x(u), \partial_j x(u)\rangle$ is the first fundamental form.
- $N(u) = \frac{\partial_1 x(u) \times \partial_2 x(u)}{\|\partial_1 x(u) \times \partial_2 x(u)\|} = G \circ x(u)$ is the Gauss map in local coordinates.
- $S_p[v_i] = DG(p)[v_i]$ is the shape operator.
- $\mathbb{I}(v_1, v_2) = \langle S_p[v_1], v_2\rangle$ is the second fundamental form.

Using Γ_{ij}^k for the Christoffel symbols and $\alpha_{ij}(u) = \mathbb{I}(\partial_i x(u), \partial_j x(u))$, we have

$$\partial_{ij} x(u) = \sum_{k=1}^{2} \Gamma_{ij}^k(u) \partial_k x(u) - \alpha_{ij}(u) N(u)$$

$$\partial_j N(u) = DG(p)[\partial_j x(u)] = \sum_{i=1}^{2} \nu_j^i(u) \partial_i x(u)$$

Third Derivatives

Gauss curvature $K(p) = \det(\nu(u))$ depends on the derivatives of N.

To this end let

$$\partial_{\ell j i} x = \sum_{k=1}^{2} \partial_\ell \Gamma_{ij}^k \partial_k x + \Gamma_{ij}^k \partial_{\ell k} x - \partial_\ell \alpha_{ij} N - \alpha_{ij} \partial_\ell N.$$

Observing that $\partial_{211} x = \partial_{121} x$, we obtain for the $\partial_2 x$-component:

$$\partial_2 \Gamma_{11}^2 + \sum_{k=1}^{2} \Gamma_{11}^k \Gamma_{2k}^2 - \alpha_{11}\nu_2^2 = \partial_1 \Gamma_{12}^2 + \sum_{k=1}^{2} \Gamma_{12}^k \Gamma_{1k}^2 - \alpha_{12}\nu_1^2$$

In other words, $\alpha_{11}\nu_2^2 - \alpha_{12}\nu_1^2$ is an intrinsic expression.

Theorema Egregium

The following expression is intrinsic:

$$\begin{aligned}
\alpha_{11}\nu_2^2 - \alpha_{12}\nu_1^2 &= \alpha_{11} \sum_{k=1}^{2} g^{2k}\alpha_{k2} - \alpha_{12} \sum_{k=1}^{2} g^{2k}\alpha_{k1} \\
&= g^{22}\left[\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\right] + g^{21}\left[\alpha_{11}\alpha_{12} - \alpha_{12}\alpha_{11}\right] \\
&= g_{11}\frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{g_{11}\cdot g_{22} - g_{12}^2} = g_{11}K
\end{aligned}$$

Theorem 1 (Theorema Egregium)**.** *The Gauss curvature K is an intrinsic feature. In particular, we have*

$$K = \frac{1}{g_{11}}\left[\left(\partial_2\Gamma_{11}^2 - \partial_1\Gamma_{12}^2\right) + \sum_{k=1}^{2}\left(\Gamma_{11}^k\Gamma_{2k}^2 - \Gamma_{12}^k\Gamma_{1k}^2\right)\right]$$

Riemann Curvature Tensor

For the Theorema Egregium, we seperated the term $\partial_{211}x - \partial_{121}x$ in an intrinsic part (using Christoffel Symbols) and an extrinsic part.

Since $\partial_{211}x = \partial_{121}x$, we were able to express the "extrinsic part" with the help of the Christoffel symbols.

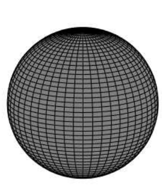
Riemann used this insight in order to define the **Riemann Curvature Tensor** R. Given two vector fields X and Y it assigns to each vector field Z and new vector field $R(X,Y)Z$. If X and Y are given as $\partial_i x$ and $\partial_j x$ of a coordinate map x, R is defined via

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z$$

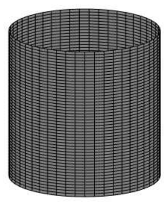
In other words, the Gauss curvature can be intrinsically written as

$$K = \frac{\langle R(\partial_2 x, \partial_1 x)\partial_1 x, \partial_2 x\rangle}{\det(g)}$$

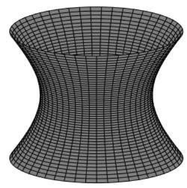
Different Gauss Curvatures



$K > 0$



$K = 0$



$K < 0$

Gauss-Bonnet

Moving Frame

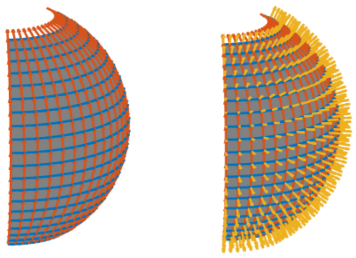
Given a coordinate map $x\colon U \to M$, the vector fields $\partial_1 x$ and $\partial_2 x$ form a base. Using Gram-Schmidt, we can create three orthonormal vector fields $Y_1, Y_2, Y_3\colon M \to \mathbb{R}^3$ via $(p = x(u))$

$$\begin{aligned} Y_1(p) &= \frac{\partial_1 x(u)}{\|\partial_1 x(u)\|} \\ Y_2(p) &= \frac{\partial_2 x(u) - \langle Y_1(p), \partial_2 x(u)\rangle}{\|\partial_2 x(u) - \langle Y_1(p), \partial_2 x(u)\rangle\|} \\ Y_3(p) &= Y_1(p) \times Y_2(p) \end{aligned}$$

We call these three vector fields a **moving frame**.

Note that a moving frame can not necessarily be derived from a coordinate map x, but it is quite usefull to have an orthonormal system at each point of the coordinate domain U respectively its codomain $x(U)$.

Moving Frame and Coordinate Maps



Parametrization induced Vector Field   Moving Frame

Derivatives of the Moving Frame

The differentials $DY_i(p)\colon T_pM \to \mathbb{R}^3$ can be written as

$$DY_i(p)[v] = \sum_{j=1}^{3} \langle \omega_{ij}(p), v\rangle Y_j(p) \qquad \omega_{ij}(p) \in T_pM$$

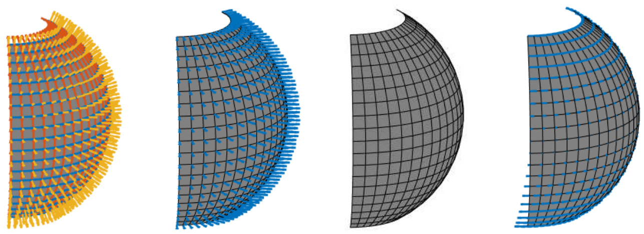
Since we have $\langle Y_i, Y_j\rangle = 0$, we obtain

$$\begin{aligned} 0 &= D\langle Y_i(p), Y_j(p)\rangle[v] = \langle DY_i(p)[v], Y_j(p)\rangle + \langle Y_i(p), DY_j(p)[v]\rangle \\ &= \langle \omega_{ij}(p) + \omega_{ji}(p), v\rangle . \end{aligned}$$

This means, we have

$$\begin{pmatrix} DY_1(p)[v] \\ DY_2(p)[v] \\ DY_3(p)[v] \end{pmatrix} = \begin{pmatrix} 0 & \langle\omega_{12}(p), v\rangle & \langle\omega_{13}(p), v\rangle \\ -\langle\omega_{12}(p), v\rangle & 0 & \langle\omega_{23}(p), v\rangle \\ -\langle\omega_{13}(p), v\rangle & -\langle\omega_{23}(p), v\rangle & 0 \end{pmatrix} \cdot \begin{pmatrix} Y_1(p) \\ Y_2(p) \\ Y_3(p) \end{pmatrix}$$

Moving Frame and Coordinate Maps



Y_1, Y_2, Y_3 $DY_1[Y_1]$ $DY_1[Y_2]$ ω_{12}

Winding Number for Circles

Given a closed non-intersecting curve $\gamma\colon I \to U$ and its corresponding curve $c = x \circ \gamma\colon I \to M$, we can define the angle function $\theta\colon c(I) \to \mathbb{R}$ via $(p = x(u) = c(t))$

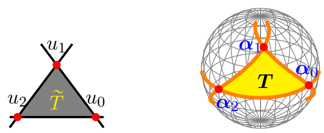
$$\theta(p) = \measuredangle(\dot c(t), Y_1(p))$$

$\theta(p)$ is unique up to multiples of 2π, but if we fix $\theta(c(0)) \in [0, 2\pi)$ there is only one unique $\theta(\cdot)$ that remains continuous.

For this setup, we have

$$\int_{c(I)} \theta'(p)\,\mathrm{d}p = \theta(c(1)) - \theta(c(0)) = 2\pi$$

Winding Number for Triangles



A triangle T can be represented by three vertices $v_0 = x(u_0)$, $v_1 = x(u_1)$, $v_2 = x(u_2) \in M$ with connected edges that can be represented as non-intersecting curves parametrized by arc-length

$$c_i\colon [0, L_i] \to M \qquad c(0) = v_i \qquad c(L_i) = v_{i\oplus 1}$$

Considering also the outer angles α_i, we obtain

$$\sum_{i<3} \int_{\operatorname{Im} c_i} \theta_i'(p)\,\mathrm{d}p + \sum_{i<3} \alpha_i = 2\pi$$

Recap: Geodesic Curvature

Given a curve $c : (0, L) \to M$ that is parametrized by arc-length, we know that $\langle \dot c(t), \ddot c(t)\rangle = 0$. Since $\frac{\nabla}{\mathrm{d}t}\dot c(p)$ contains the component of $\ddot c(t)$ in T_pM, we have

$$\frac{\nabla}{\mathrm{d}t}\dot c(p) = \kappa_g(p)\dot c(t)^{\perp},$$

where $\dot c(t)^{\perp}$ is the vector in T_pM that is normal to $\dot c(t)$.

Using the angle function θ, we obtain

$$\kappa_g(p) = \left\langle \frac{\nabla}{\mathrm{d}t}\dot c(p), \dot c(t)^{\perp} \right\rangle$$

with

$$\begin{pmatrix} \dot c(t) \\ \dot c(t)^{\perp} \end{pmatrix} = \begin{pmatrix} \cos(\theta(p)) & \sin(\theta(p)) \\ -\sin(\theta(p)) & \cos(\theta(p)) \end{pmatrix} \cdot \begin{pmatrix} Y_1(p) \\ Y_2(p) \end{pmatrix}$$

Integrating Geodesic Curvature

We have

$$\nabla_{\dot c}\dot c = \cos(\theta)\nabla_{\dot c}Y_1 + \sin(\theta)\nabla_{\dot c}Y_2 - \sin(\theta)\theta' Y_1 + \cos(\theta)\theta' Y_2$$

and therefore

$$\begin{aligned}\kappa_g &= \langle \nabla_{\dot c}\dot c, -\sin(\theta)Y_1 + \cos(\theta)Y_2\rangle \\ &= \theta' + \langle \cos(\theta)\nabla_{\dot c}Y_1 + \sin(\theta)\nabla_{\dot c}Y_2, -\sin(\theta)Y_1 + \cos(\theta)Y_2\rangle \\ &= \theta' + \langle \omega_{12}, \dot c\rangle\end{aligned}$$

In other words

$$-\sum_{i<3}\int_{\operatorname{Im} c_i}\langle \omega_{12}(p), \dot c_i(t)\rangle \,\mathrm{dp} + \sum_{i<3}\int_{\operatorname{Im} c_i}\kappa_g(p)\,\mathrm{dp} + \sum_{i<3}\alpha_i = 2\pi$$

Integration Theorem of Green

Given a vector field $V\colon \mathbb{R}^2 \to \mathbb{R}^2$, and $S = [0,1]\times[0,1]$ the **unit square**. Then we have

$$\int_S \partial_1 V^2(x,y) - \partial_2 V^1(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 V^2(1,y) - V^2(0,y)\,\mathrm{d}y + \int_0^1 V^1(x,0) - V^1(x,1)\,\mathrm{d}x$$

In other words, we have $\int_S \partial_1 V^2(p) - \partial_2 V^1(p)\,\mathrm{dp} = \int_{\partial S} V^1\,\mathrm{d}x + V^2\,\mathrm{d}y$.

This is the **Green's theorem**, which is also true for general open sets S with a smooth boundary ∂S parametrized by $c\colon [0;1] \to \partial S$:

$$\int_S \partial_1 V^2(p) - \partial_2 V^1(p)\,\mathrm{dp} = \int_{\partial S} V^1\,\mathrm{d}x + V^2\,\mathrm{d}y = \int_0^1 \langle V(c(t)), \dot c(t)\rangle\,\mathrm{dt}$$

Integrating ω_{12}

In order to express the integral of $\langle \omega_{12}, \dot c\rangle$ in means of p alone, we have

$$\int_{\operatorname{Im} c_i}\langle \omega_{12}(p), \dot c(t)\rangle\,\mathrm{dp} = \int_{\operatorname{Im} c_i}\langle \nabla_{\dot c}Y_1(p), Y_2(p)\rangle\,\mathrm{dp}$$

Analogously to the Green integration theorem, one can show that

$$\int_{\partial T}\langle \nabla_{\dot c}Y_1(p), Y_2(p)\rangle\,\mathrm{dp} = \int_T \langle R(Y_1,Y_2)Y_1(p), Y_2(p)\rangle\,\mathrm{dp} = -\int_T K(p)\,\mathrm{dp}$$

In other words the **Theorem of Gauss-Bonnet for Triangles** is

$$\int_T K(p)\,\mathrm{dp} + \int_{\partial T}\kappa_g(p)\,\mathrm{dp} + \sum_{i<3}\alpha_i = 2\pi$$

Gauss-Bonnet without Boundary

Let us assume we have a smooth triangulation of a closed surface M that uses the vertex set V, the edge set E and the face set F, then we have

$$\begin{aligned}2\pi\cdot|F| &= \sum_{T\in F}\left[\int_T K(p)\,\mathrm{dp} + \int_{\partial T}\kappa_g(p)\,\mathrm{dp} + \sum_{i<3}\alpha_i^{(T)}\right] \\ &= \int_M K(p)\,\mathrm{dp} + |E|\cdot 2\pi - |V|\cdot 2\pi\end{aligned}$$

In other words the **Theorem of Gauss-Bonnet for Closed Surfaces** is

$$\int_M K(p)\,\mathrm{dp} = 2\pi\,(|F| - |E| + |V|)$$

Gauss-Bonnet with Smooth Boundary

Let us assume we have a surface M with a smooth boundary. Further assume a smooth triangulation that uses the vertex set V, the edge set E and the face set F. Then we have

$$\begin{aligned}2\pi\cdot|F| &= \sum_{T\in F}\left[\int_T K(p)\,\mathrm{dp} + \int_{\partial T}\kappa_g(p)\,\mathrm{dp} + \sum_{i<3}\alpha_i^{(T)}\right] \\ &= \int_M K(p)\,\mathrm{dp} + \int_{\partial M}\kappa_g(p)\,\mathrm{dp} + |E|\cdot 2\pi - |V|\cdot 2\pi\end{aligned}$$

In other words the **Theorem of Gauss-Bonnet for Surfaces With Smooth Boundaries** is

$$\int_M K(p)\,\mathrm{dp} + \int_{\partial M}\kappa_g(p)\,\mathrm{dp} = 2\pi\,(|F| - |E| + |V|)$$

Euler Characteristic

Euler Chracteristic

Given a triangulation (V,E,F) of a surface M, we call

$$\chi(M) = |V| - |E| + |F| \in \mathbb{Z}$$

the **Euler Characteristic** of M.

Due to the Gauss-Bonnet theorem, we know that

$$\chi(M) = \frac{\int_M K(p)\,\mathrm{dp} + \int_{\partial M}\kappa_g(p)\,\mathrm{dp}}{2\pi}$$

is a global property of M.

For every triangulation (V,E,F) of $\mathbb{S}^2$ we have

$$|V| - |E| + |F| = \chi(\mathbb{S}^2) = 2$$

Gauss Curvature at a Vertex

Given a discrete triangulation (V,E,F) of a surface M, we assume that at a vertex $v\in V$, we have k triangles $T_1,\dots,T_k$ with the angles α_i, β_i and γ_i (γ_i at v). It is common to use the following approximation of the Gauss curvature as a feature (point descriptor)

$$K(v) := \frac{\int_{\bigcup_{i=1}^k \frac{1}{3}T_i} K(p)\,\mathrm{dp}}{\sum_{i=1}^k \frac{1}{3}\operatorname{area}(T_i)} \approx \frac{2\pi - \sum_{i=1}^k \gamma_i}{\sum_{i=1}^k \frac{1}{3}\operatorname{area}(T_i)}$$